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Shadow multiplets in AdS_4/CFT_3 and the super-Higgs mechanism †

M. Billó[1], D. Fabbri[1], P. Fré[1], P. Merlatti[1] and A. Zaffaroni[2]

[1] *Dipartimento di Fisica Teorica, Universitá di Torino,*
via P. Giuria 1, I-10125 Torino,
and
Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy
[2] *Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Milano, Italy*

Abstract

We discuss a general pairing that occurs in compactifications of M–theory on $\mathrm{AdS}_4 \times X^7$ backgrounds between massless ultra short multiplets and their massive shadows, namely certain universal long multiplets with fixed protected dimensions. In particular we consider the shadow of the short graviton multiplet in $\mathcal{N} = 3$ compactifications. It turns out to be a massive spin $\frac{3}{2}$ multiplet with scale dimension $E_0 = 3$ and with the quantum numbers of a superHiggs multiplet. Hence each $\mathcal{N} = 3$ $\mathrm{AdS}_4 \times X^7$ vacuum is actually to be interpreted as a spontaneously broken phase of an $\mathcal{N} = 4$ theory. Comparison with standard gauged $\mathcal{N} = 4$ supergravity in 4 dimensions reveals the unexpected bound $E_0 < 3$ on the dimension of the broken gravitino multiplet. This hints to the existence of new versions of extended supergravities, in particular $\mathcal{N} = 4$ where such upper bounds are evaded and where all possible vacua have a reduced supersymmetry $\mathcal{N}_0 < \mathcal{N}$. We name them shadow supergravities. In particular, using arguments based on the solvable Lie algebra parametrization of the scalar manifold, we discuss the possible structure of shadow $\mathcal{N} = 4$ supergravity. Using our previous results on the SCFT dual of the $\mathrm{AdS}_4 \times N^{0,1,0}$ vacuum we discuss the SCFT realization of the universal $\mathcal{N} = 3$ shadow multiplet. RG flows from an $\mathcal{N} = 4$ to an $\mathcal{N} = 3$ phase are ruled out by the fact that the $\mathcal{N} = 4$ vacuum is at infinite distance in moduli space, denoting the presence of a topology change.

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1 Introduction

The study of Anti-de-Sitter compactifications of M-theory or type II string theories is a fascinating subject, resurrected to new life after the Maldacena conjecture [1]. The profound interplay between AdS compactifications and conformal field theories has led to a new understanding of several results derived in the eighties in the context of the Kaluza-Klein program. In this paper, motivated by field theory considerations, we shall discuss some general features of Freund-Rubin compactifications that were previously overlooked and we are led to some unexpected and stimulating conclusions about the existence of so far unknown aspects of both supergravity and superconformal field theories. Shadowing of long multiplets behind short ones in Freund-Rubin compactifications, and shadow supergravities that have not been constructed but whose existence is suggested by many considerations, are the main subject of this paper. In parallel, we shall discuss implications of our results for CFT's.

The natural arena for our discussion are M-theory or Type II string backgrounds of the form $\mathrm{AdS}_{p+2} \times X^{d-p-2}$ in d-dimensions (where X_{d-p-2} is an Einstein manifold), and the associated dual CFT's. Even if we mainly discuss the case of AdS_4, some of our results apply to AdS_5 as well with minor modifications. In previous papers [2, 3], we proposed candidate dual CFT's for some supersymmetric AdS-compactification of M-theory on coset spaces, discussing the $\mathcal{N} = 2$ cases $Q^{1,1,1}$ and $M^{1,1,1}$ and the $\mathcal{N} = 3$ solution $N^{0,1,0}$. The remaining supersymmetric case $V^{5,2}$ has been discussed in [4]. In all these cases, as well as in the case of the firstly discussed type IIB compactification $\mathrm{AdS}_5 \times T^{1,1}$ [5, 6, 7, 8], the comparison between the KK spectrum and the spectrum of conformal composite operators of the CFT is completely successful and the analysis of baryonic operators corresponding to wrapped 5-branes gives results in agreement with quantum field theory expectations. The analysis of the spectrum reveals that all these compactifications share common features. One notably one, firstly noticed in [6], is the existence of long multiplets with protected rational dimensions. Following a suggestion in [8], these multiplets can often be written in the CFT as tensor product of short and massless multiplets, so that they can be identified on the quantum field theory side. Their protected dimension suggests the existence of some non-renormalization theorem.

The first purpose of this paper is to illustrate an intriguing general feature of Kaluza Klein spectra for compactifications of type $\mathrm{AdS}_4 \times X_7$ when some $\mathcal{N}-$extended supersymmetry is present. As a consequence of the mass formulae derived in the eighties [9], there is a symmetry structure that so far was not appropriately noticed and explored. Indeed, the supermultiplets are paired by curious relations that to each multiplet associate another *shadow* multiplet of different spin and of different type, but with masses exactly predicted by those of the parent multiplet. The shadows of short multiplets are generically long ones, but with rational conformal dimensions and explain why the appearance of protected multiplets is so common is AdS-compactifications on Einstein manifolds.

Of special interest is the long shadow multiplet of the massless graviton. Such shadow multiplet is *universal* since it has the volume of the internal manifold as one of its scalar component and, moreover, it has a *universal* structure that is independent from the detailed geometry of X_7. In $\mathcal{N} = 2$ compactifications, the massless graviton shadow is a long, massive, *vector* multiplet in the same R–symmetry representation as the graviphoton. In $\mathcal{N} = 3$ compactifications, instead, the shadow of the massless graviton multiplet is a long, massive *gravitino*. In all Freund-Rubin compactification, the mass of the scalar associated with the volume (which is paired to the massless graviton) is such that the corresponding CFT operator has dimension 6. This is closely reminiscent of a similar phenomena in type II AdS_5 compactifications, where the volume of the internal manifold corresponds to the CFT operator F^4, with dimension 8, which is known to satisfy some non-renormalization theorem.

We shall be mainly concerned with the $\mathcal{N} = 3$ compactification on $N^{0,1,0}$, where all the previous phenomena can be also discussed from a different perspective. All the components of the long gravitino multiplet are constructed with *constant* harmonics, suggesting the existence of a consistent truncation to a theory with only a massless graviton and a massive gravitino. This fact is extremely intriguing since it seems to indicate that the $\mathcal{N} = 3$ compactification of M–theory should admit an interpretation as a spontaneously broken phase of some suitable $\mathcal{N} = 4$ theory. Exploring this possibility is the main focus of the second part of this present paper, where we analyze the theory from the perspective of four-dimensional gauged supergravity. We shall look for an $\mathcal{N} = 4$ gauged supergravity with an $\mathcal{N} = 3$ critical point containing a massless graviton and a massive broken gravitino with the quantum numbers corresponding to those of the $N^{0,1,0}$ compactification. We would like to stress that the analysis of the low-energy supergravity and its critical points is at the basis of the study of the deformations of the dual CFT and the induced RG flow [10]. In this context, using older results in [11], the phase space and the RG trajectories for the $N^{0,1,0}$ compactification have been partially studied in [12], without looking for $\mathcal{N} = 4$ supersymmetry. We shall complete the reduction of the Lagrangian by adding the scalars corresponding to the internal photon, not discussed in [11, 12], and we shall compared with the known version of $\mathcal{N} = 4$ gauged supergravities [13]. Here we shall discover some intriguing facts.

The existing versions of $\mathcal{N} = 4$ gauged supergravity exhibit the spontaneous symmetry breaking $\mathcal{N} = 4 \to \mathcal{N} = 3$ that we are looking for [14]. This was indeed the original motivation for this paper. Perhaps more surprising, there is a continuum of $\mathcal{N} = 3$ critical points, with arbitrary value for mass of the gravitino but strictly below the predicted value for $N^{0,1,0}$. The $N^{0,1,0}$ critical point can be realized only at the boundary of the moduli space, at infinite distance from the $\mathcal{N} = 4$ point. The interpolation between the $N^{0,1,0}$ compactification and its parent $\mathcal{N} = 4$ theory necessarily involves some change of topology.

From the point of view of four-dimensional supergravity, this result is intriguing and suggests the existence of overlooked supergravities. The unitarity bound for

the E_0 (dimension of the operator dual to the lowest component) quantum number for a gravitino multiplet is $E_0 > 1$. In the known supergravities, there is a variety of $\mathcal{N} = 3$ theories with all possible values $E_0 < 3$. $E_0 = 3$ is, of course, the value for $N^{0,1,0}$ and for the *universal* volume multiplet in Freund-Rubin compactifications. There is no theoretical reason for an upper bound on E_0. We suggest that four-dimensional supergravities are missing where the values $E_0 > 3$ are realized. We shall argue that these supergravities have been overlooked in the past. The reason is that they should admit $\mathcal{N} = 4$ supersymmetry but no $\mathcal{N} = 4$ vacuum, leaving the freedom of relaxing the constraints imposed by global symmetries. We shall give some arguments based on the coset structure of the scalar manifold and analogies with similar situations with missing BPS states in supergravity theories [15].

Finally, we should mention that we expect that there are certainly many other results concerning CFT's and RG flows, following from our supergravity analysis, than those presented in this paper. We leave them for future work.

The paper is organized as follows. Section 2 fixes the conventions for Freund-Rubin compactifications. Section 3 deals with the explicit construction of the shadow supermultiplets, with particular emphasis for the case $N^{0,1,0}$. Section 4 discusses the $\mathcal{N} = 3$ critical points of four-dimensional supergravity, the explicit reduction from 11 to 4 dimensions and the upper bound on E_0. Section 5, which is the conjectural part of this paper, suggests and motivate the search for new shadow supergravities. Section 6 presents for completeness the explicit form of the long gravitino multiplet of $N^{0,1,0}$, using results obtained in [3]. Section 7 contains some conclusions and a short discussion of the CFT interpretation of our supergravity results. Finally, the appendices contain conventions and useful formulae.

2 Freund–Rubin compactifications of 11D supergravity

For $D = 11$ supergravity [16, 17] our basic conventions are those of the geometric formulation [17] and for Kaluza Klein compactifications we follow the conventions of [18, 19, 9] (see appendix A). The bosonic action of 11D supergravity reads:

$$S = \frac{1}{\kappa_{11}^2}\int \mathcal{R}\det V - \frac{1}{16\kappa_{11}^2}\int F\wedge {}^*F - \frac{1}{96\kappa_{11}^2}\int F\wedge F\wedge A\ , \tag{2.1}$$

where $\mathcal{R}$ is the scalar curvature, V^M $(M = 0, \dots 10)$ are the vielbein, A is a three-form and F its four-form field-strength.

The equations of motion that follow from eq. (2.1) are

$$\begin{aligned} \mathcal{R}^M_{\ N} &= 6F^{MP_1P_2P_3}F_{NP_1P_2P_3} - \frac{1}{2}\delta^M_{\ N}F^2\ ; & (2.2)\\ D_M F^{MP_1P_2P_3} &= \frac{1}{96}\varepsilon^{P_1P_2P_3N_1\dots P_8}F_{N_1\dots N_4}F_{N_5\dots N_8} & (2.3)\end{aligned}$$

where $\mathcal{R}^M_N$ is the Ricci tensor. Through all the paper, we use "flat" indices, namely we write all tensor components with respect to the vielbein basis.

The supersymmetry transformation of the gravitino ψ_M is

$$\delta_\epsilon \psi_M = D_M \epsilon - \left(\frac{\mathrm{i}}{3} F_{P_1 P_2 P_3 M} \Gamma^{P_1 P_2 P_3} - \frac{\mathrm{i}}{8} F^{P_1 P_2 P_3 P_4} \Gamma_{P_1 P_2 P_3 P_4 M} \right) \epsilon \ . \tag{2.4}$$

Freund-Rubin compactifications Freund-Rubin [20] (FR) compactifications are solutions of the field equations of $11D$ supergravity (2.2) in which the spacetime $\mathcal{M}_{11}$ has the factorized form

$$\mathcal{M}_{11} = \mathrm{AdS}_4 \times X^7 \ , \tag{2.5}$$

the relevant one for investigating the AdS/CFT correspondence between $3D$ superconformal gauge theories on the $M2$–brane world volume and supergravity in $4D$ anti de Sitter space. The only non-vanishing components of F are the 4-dimensional ones:

$$F_{abcd} = e \, \varepsilon_{abcd} \ . \tag{2.6}$$

The parameter e sets the scale for both the 4-dimensional and the 7-dimensional cosmological constant (also X^7 must be an Einstein space):

$$\mathcal{R}_{ab} = -24 \, e^2 \, \eta_{ab} \ , \qquad \mathcal{R}_{\alpha\beta} = 12 \, e^2 \, \eta_{\alpha\beta} \ . \tag{2.7}$$

Greek letters $\alpha, \beta, \ldots$ will always be reserved to flat 7-dimensional indices, while Latin letters $a, b \ldots$ will stand for 4-dimensional flat indices. We denote by $B^{\alpha\beta}$ the spin connection one-form, and by B^α the vielbein on X^7. Furthermore we decompose the 11-dimensional gamma matrices (Γ^M) as tensor products of 4-dimensional (γ^a) and 7-dimensional ones (τ^α) as follows:

$$\Gamma^a = \gamma^a \times \mathbf{1}_7 \ , \qquad \Gamma^\alpha = \gamma^5 \times \tau^\alpha \ . \tag{2.8}$$

In the Freund–Rubin background the gravitino is set to zero; we can search for preserved supersymmetries by requiring that its variation vanishes as well. According to (2.5) we write the 11-dimensional spinor parameter $\epsilon(x, y)$ as the tensor product of a 4- and a 7-dimensional spinor, $\epsilon(x, y) = \epsilon(x) \times \eta(y)$. To every independent 7-dimensional (commuting) spinor parameter $\eta(y)$ that satisfies

$$D\eta(y) \equiv (d - \frac{1}{4} B^{\alpha\beta} \tau_{\alpha\beta}) \eta(y) = e \, B^\alpha \tau_\alpha \eta(y) \tag{2.9}$$

there is associated a residual supersymmetry in the AdS_4 space.

Indeed the vanishing of (2.4) for the gravitino $\psi_a(x)$ in the background (2.6) reduces to the vanishing of its supersymmetry transformation in AdS_4:

$$D^{(\mathrm{AdS})}_a \epsilon(x) \equiv (\partial_a - \frac{1}{4} \omega^{bc}{}_a \gamma_{bc} - 2 \, e \, \gamma_5 \gamma_a) \epsilon(x) = 0 \ , \tag{2.10}$$

whose integrability is guaranteed by the expression of the AdS_4 curvature, $R^{ab}{}_{cd} = -16 \, e^2 \, \delta^{ab}_{cd}$, that corresponds to the Ricci tensor (2.7).

3 Shadow multiplets in Kaluza Klein theory

Consider a generic Freund–Rubin compactification of M–theory on $\mathrm{AdS}_4 \times X^7$. The fluctuations of the eleven–dimensional fields around such a background can be expanded into harmonics on the compact 7–manifold and the linearized field equations can be suitably diagonalized into eigenmodes of definite mass. The resulting general formulae were derived in [19, 21] and organized into a systematic way in [9].

In the present section we deal with an intriguing general feature of such Kaluza Klein spectra when some $\mathcal{N}$–extended supersymmetry is present. In that case all states organize themselves into supermultiplets of the relevant supersymmetry algebra namely $\mathrm{Osp}(\mathcal{N}|4)$. However this is not the end of the story. As a consequence of the results of [9], there is a further symmetry structure that so far was not appropriately noticed and explored. Indeed, going beyond the implications of pure superalgebra representation theory, the supermultiplets are further paired by curious relations that to each multiplet associate another *shadow* multiplet of different spin and of different type, but with masses fixed in terms of those of the parent multiplet.

In general, the shadows of short multiplets are long ones. Yet, because of the shadowing relation their conformal dimensions, derived from those of the parent multiplets, are rational. Hence the appearance of rational long multiplets [2, 8, 22] is partially explained by the shadowing relation.

Of special interest are the long shadow multiplets of massless ultrashort multiplets. Such shadow multiplets have a *universal* structure that is independent from the detailed geometry of X^7 and simply follows from the structure of the $D = 11$ field equations plus the constraints of supersymmetry.

In $\mathcal{N} = 2$ compactifications, the massless graviton multiplet has always a shadow which is a long, massive, *vector* multiplet in the same R–symmetry representation as the graviphoton. In $\mathcal{N} = 3$ compactifications, instead, the shadow of the massless graviton multiplet is a long, massive *gravitino.* This fact is of utmost interest since it seems to indicate that any $\mathcal{N} = 3$ compactification of M–theory on $\mathrm{AdS}_4 \times X^7$ should admit an interpretation as a spontaneously broken phase of some suitable $\mathcal{N} = 4$ theory. Exploring this possibility is the main focus of the present paper and will lead to some unexpected and stimulating conclusions hinting to the existence of so far unknown aspects of both supergravity and superconformal field theories.

Also massless vector multiplets have interesting shadows but, both in $\mathcal{N} = 2$ and in $\mathcal{N} = 3$ compactifications, these shadows are long, rational, *graviton* multiplets. This excludes an interpretation as Higgs or super-Higgs phenomena. The same is true of compactifications with $\mathcal{N} > 4$, like the maximally extended case of $\mathrm{AdS}_4 \times S^7$. Here the shadow multiplets of the massless graviton multiplet are all massive short graviton multiplets since the entire spectrum is composed of short multiplets [23]. Obviously there is no super-Higgs mechanism in such cases.

In section 3.1 we illustrate the general mechanism of shadow multiplet generation. Then in section 3.2 we focus on the special $\mathcal{N} = 3$ case and we discuss the

universal structure of the massive gravitino multiplet generated by the shadow of the $\mathcal{N} = 3$ massless graviton multiplet.

3.1 Shadow multiplet generation

Since our argument on the existence of shadow multiplets relies heavily on the Fermi–Bose harmonic relations constructed in [9], we have to recall the notations and the main results of that paper.

It is possible (see eq.s (3.21) of [9]) to parametrize the Kaluza Klein expansion of the field fluctuations in terms of 4-dimensional fields that are mass eigenstates. For the fluctuations h_{MN} of the metric one sets

$$\begin{aligned} h_{ab}\,(x,y) &= \Big(h^I_{ab}\,(x) - \frac{3}{M_{(0)^3}+32} D_{(a}D_{b)}\Big[(2+\sqrt{M_{(0)^3}+36}\,)S^I\,(x) \\ &+ (2-\sqrt{M_{(0)^3}+36}\,)\Sigma^I\,(x)\Big] + \tfrac{5}{4}\delta_{ab}\Big[(6-\sqrt{M_{(0)^3}+36}\,)S^I\,(x) \\ &+ (6+\sqrt{M_{(0)^3}+36}\,)\Sigma^I\,(x)\Big]\Big)\,Y^I\,(y)\ , \end{aligned} \tag{3.1}$$

$$\begin{aligned} h_{a\beta}\,(x,y) &= [(\sqrt{M_{(1)(0)^2}+16}-4)A^I_a\,(x) \\ + &+ (\sqrt{M_{(1)(0)^2}+16}+4)W^I_a\,(x)\,]\,Y^I_\beta\,(y)\ , \end{aligned} \tag{3.2}$$

$$\begin{aligned} h_{\alpha\beta}\,(x,y) &= \phi^I\,(x)\,Y^I_{(\alpha\beta)}\,(y) - \delta_{\alpha\beta}\Big[(6-\sqrt{M_{(0)^3}+36})S^I\,(x) \\ &+ (6+\sqrt{M_{(0)^3}+36})\Sigma^I\,(x)\Big]\ Y^I\,(y)\ . \end{aligned} \tag{3.3}$$

For the fluctuations a_{MNR} of the three form field, one has

$$\begin{aligned} a_{abc}\,(x,y) &= 2\,\varepsilon_{abcd}\,D_d(S^I\,(x)+\Sigma^I\,(x))Y^I\,(y)\ , \\ a_{ab\gamma}\,(x,y) &= \tfrac{2}{3}\,\varepsilon_{abcd}\,(D_cA^I_d\,(x)+D_cW^I_d\,(x))\,Y^I_\gamma\,(y)\ , \\ a_{a\beta\gamma}\,(x,y) &= Z^I_a\,(x)\,Y^I_{[\beta\gamma]}\,(y)\ , \end{aligned} \tag{3.4}$$

$$a_{\alpha\beta\gamma}\,(x,y) = \pi^I\,(x)\,Y^I_{[\alpha\beta\gamma]}\,(y)\ . \tag{3.5}$$

Finally, for the fluctuations of the gravitino field,

$$\begin{aligned} \psi_a\,(x,y) &= \Big(\chi^I_a\,(x) + \frac{\frac{4}{7}M_{(1/2)^3}+8}{M_{(1/2)^3}+8}[D_a\lambda^I_L\,(x)\,]_{3/2} \\ &- (6+\tfrac{3}{7}M_{(1/2)^3})\gamma_5\gamma_a\lambda^I_L\,(x)\Big)\,\Xi^I\,(y)\ , \end{aligned} \tag{3.6}$$

$$\psi_\alpha(x,y) = \lambda^I_T\,(x)\,\Xi^I_\alpha\,(y) + \lambda^I_L\,(x)\,[\nabla_\alpha\Xi^I\,(y)\,]_{3/2}\ . \tag{3.7}$$

We denote by x the coordinates of four dimensional space, while y are the coordinates on the compact 7–manifold X^7.

For the harmonics on X^7 that appear in the KK expansion (3.1-3.6) we have followed the conventions of [9]. These conventions are summarized in Table 1, and

Irrep	Dimension	Operator	Harmonic	Eigenvalue
$[0,0,0]$	$\mathbf{1}$	$\boxtimes_{(0)^3}$	Y^I	$M_{(0)^3}$
$[1,0,0]$	$\mathbf{7}$	$\boxtimes_{(1)(0)^2}$	Y^I_α	$M_{(1)(0)^2}$
$[1,1,0]$	$\mathbf{21}$	$\boxtimes_{(1)^2(0)}$	$Y^I_{[\alpha\beta]}$	$M_{(1)^2(0)}$
$[1,1,1]$	$\mathbf{35}$	$\boxtimes_{(1)^3}$	$Y^I_{[\alpha\beta\gamma]}$	$M_{(1)^3}$
$[2,0,0]$	$\mathbf{27}$	$\boxtimes_{(2)(0)^2}$	$Y^I_{(\alpha\beta)}$	$M_{(2)(0)^2}$
$\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]$	$\mathbf{8}$	$\boxtimes_{(\frac{1}{2})^3}$	Ξ^I	$M_{(\frac{1}{2})^3}$
$\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]$	$\mathbf{48}$	$\boxtimes_{(\frac{3}{2})(\frac{1}{2})^2}$	Ξ^I_α	$M_{(\frac{3}{2})(\frac{1}{2})^2}$

Table 1: Conventions for the harmonics on X^7.

consist in the following. The harmonics on X^7 are grouped into seven infinite towers corresponding to as many irreducible representations of the tangent group SO(7) that appear in the decomposition $4 \oplus 7$ of eleven dimensional tensors and spinors. Since SO(7) has rank 3, its *irreps* are labeled by three numbers $[\lambda_1, \lambda_2, \lambda_3]$ that we take to be the Young labels[1].

The explicit form of the invariant operators appearing in table (1) is given below:
- Laplacian on scalars:

$$\boxtimes_{(0)^3} Y_\alpha \equiv D^\mu D_\mu Y = \Box Y_\alpha = M_{(0)^3} Y_\alpha ; \tag{3.8}$$

- Hodge–de Rham operator on one–forms:

$$\boxtimes_{(1)(0)^2} Y_\alpha \equiv 2 D^\mu D_{[\mu} Y_{\alpha]} = \left(\Box + 24 e^2\right) Y_\alpha = M_{(1)(0)^2} Y_\alpha ; \tag{3.9}$$

- Hodge–de Rham operator on two–forms:

$$\boxtimes_{(1)^2 0} Y_{[\alpha\beta]} \equiv 3 D^\mu D_{[\mu} Y_{\alpha\beta]} = \left[\left(\Box + 40 e^2\right) \delta^{\lambda\mu}_{\alpha\beta} - 2 C_{\alpha\beta}{}^{\lambda\mu}\right] Y_{[\lambda\mu]} = M_{(1)^2 0} Y_{[\alpha\beta]} ; \tag{3.10}$$

- $*d$ operator on three forms:

$$\boxtimes_{(1)^3} Y_{[\alpha\beta\gamma]} \equiv \frac{1}{24}\epsilon_{\alpha\beta\gamma\lambda\mu\nu\rho} D^\lambda Y^{\mu\nu\rho} \equiv \star\, dY_{[\alpha\beta\gamma]} = M_{(1)^3} Y_{[\alpha\beta\gamma]} ; \tag{3.11}$$

[1] That is, for bosonic tensors with symmetry represented by a Young tableaux, λ_i is the number of boxes in the i-th row of the tableaux. For gamma–traceless irreducible spinor tensors λ^i is 1/2 plus the number of boxes.

- Lichnerowitz operator on symmetric tensors:

$$\boxtimes_{(2)(0)^2} Y_{(\alpha\beta)} = \left[\left(\Box + 40\, e^2 \right) \delta^{(\lambda\mu)}_{(\alpha\beta)} - 4\, C_{(\alpha\beta)}{}^{(\lambda\mu)} \right] Y_{(\lambda\mu)} \;=\; M_{(2)(0)^2} Y_{(\alpha\beta)} \;; \tag{3.12}$$

- "de Sitter" Dirac operator:

$$\boxtimes_{(\frac{1}{2})^3} \Xi \equiv \tau^\mu \nabla_\mu \Xi \;=\; \tau^\mu \left(D_\mu - e\, \tau_\mu \right) \Xi \;=\; (\not{D} - 7\, e\,) \,\Xi \;=\; M_{(\frac{1}{2})^3} \Xi \;; \tag{3.13}$$

- "de Sitter" Rarita Schwinger operator:

$$\boxtimes_{(\frac{3}{2})(\frac{1}{2})^2} \Xi_\alpha \;\equiv\; \tau_{\alpha\mu\nu} \nabla^\mu \Xi^\nu - \tfrac{5}{7} \tau_\alpha \tau^{\mu\nu} \nabla_\mu \Xi_\nu \;=\; (\not{D} - 5\, e\,) \,\Xi_\alpha \;=\; M_{(\frac{3}{2})(\frac{1}{2})^2} \Xi_\alpha \,, \tag{3.14}$$

where the conventions for 7 geometry are those of [9] (see Appendix A). In particular,

$$\mathcal{C}^{\alpha\beta}{}_{\mu\nu} = \mathcal{R}^{\alpha\beta}{}_{\mu\nu} - 4e^2\, \delta^{\alpha\beta}_{\mu\nu} \tag{3.15}$$

is the Weyl tensor on the Einstein manifold X^7, with Ricci tensor as in eq. (2.7)

Measuring everything in units of the Freund–Rubin scale, *i.e.* setting $e = 1$, the masses of the mass eigenstates appearing in the Kaluza Klein expansion (eq.s (3.1-3.6)) are expressed in terms of the eigenvalues of the appropriate operators as follows:

$$m_h^2 \;=\; M_{(0)^3} \,, \tag{3.16}$$

$$m_\Sigma^2 \;=\; M_{(0)^3} + 176 + 24\sqrt{M_{(0)^3} + 36} \,, \tag{3.17}$$

$$m_S^2 \;=\; M_{(0)^3} + 176 - 24\sqrt{M_{(0)^3} + 36} \,, \tag{3.18}$$

$$m_\phi^2 \;=\; M_{(2)(0)^2} \,, \tag{3.19}$$

$$m_\pi^2 \;=\; 16 \left(M_{(1)^3} - 2 \right) \left(M_{(1)^3} - 1 \right) \,, \tag{3.20}$$

$$m_W^2 \;=\; M_{(1)(0)^2} + 48 + 12\sqrt{M_{(1)(0)^2} + 16} \,, \tag{3.21}$$

$$m_A^2 \;=\; M_{(1)(0)^2} + 48 - 12\sqrt{M_{(1)(0)^2} + 16} \,, \tag{3.22}$$

$$m_Z^2 \;=\; M_{(1)^2(0)} \,, \tag{3.23}$$

$$m_{\lambda_L} \;=\; - \left(M_{(\frac{1}{2})^3} + 16 \right) \,, \tag{3.24}$$

$$m_{\lambda_T} \;=\; M_{(\frac{3}{2})(\frac{1}{2})^2} + 8 \,, \tag{3.25}$$

$$m_\chi \;=\; M_{(\frac{1}{2})^3} \,. \tag{3.26}$$

Each mass eigenmode corresponds to an irreducible unitary representation (UIR) of the anti de Sitter group $\mathrm{SO}(2,3) \sim \mathrm{Sp}(4,\mathbb{R})$ which is labeled by two weights, the $\mathrm{SO}(3)$ spin s (which is $s = 0$ for scalars, $s = \frac{1}{2}$ for spinors and so on up to $s = 2$ for

massive gravitons) and the energy eigenvalue E which satisfies the unitarity lower bound [24, 25, 26]:

$$E \geq s+1\,. \tag{3.27}$$

In the alternative three-dimensional conformal interpretation of the $\mathrm{SO}(2,3)$ group each UIR is a primary conformal field with scale dimension $\Delta = E$ and Lorentz $\mathrm{SO}(1,2)$ spin s (compare with [28] for further details).

In [9] the relations between the masses (3.16 –3.26) and the conformal dimensions are given in the following form:

$$\begin{aligned} m^2_{(0)} &= 16\left(E_{(0)}-2\right)\left(E_{(0)}-1\right)\,, \\ |m_{(\frac{1}{2})}| &= 4E_{(\frac{1}{2})}-6\,, \\ m^2_{(1)} &= 16\left(E_{(1)}-2\right)\left(E_{(1)}-1\right)\,, \\ |m_{(\frac{3}{2})}+4| &= 4E_{(\frac{3}{2})}-6\,, \end{aligned} \tag{3.28}$$

where $m_{(s)}$ and $E_{(s)}$ denote the mass and energy of a field of spin s [2].

Supersymmetry and shadow multiplets Let us consider unitary irreducible representations of the superalgebra $\mathrm{Osp}(\mathcal{N}|4)$, *i.e.* the supersymmetric extension of $\mathrm{SO}(2,3)$ with $\mathcal{N}$ supercharges. Each of them is a supermultiplet, represented by a suitable constrained superfield, that contains fields whose spins s and dimensions E are related to each other. Via eq.s (3.16,3.26,3.28), such relations translate into relations on the eigenvalues of the various Laplace Beltrami operators (3.8,3.14).

These relations hint to a sort of mirror image of the supersymmetry algebra that is realized on the internal compact manifold X^7. This idea was thoroughly analyzed in [9] and traced back to the existence of (commuting) Killing spinors η^A ($A = 1,\ldots\mathcal{N}$) satisfying the Killing spinor equation eq. (2.9), where $\mathcal{N}$ is the number of preserved supersymmetries of the $\mathrm{AdS}_4 \times X^7$ compactification one considers. By means of the Killing spinors η^A, to each harmonic Y that is an eigenstate of a bosonic Laplacian $\boxtimes_{n_1,n_2,n_3}$ one can associate another fermionic harmonic Ξ that is an eigenstate of a fermionic laplacian $\boxtimes_{n_1\pm\frac{1}{2},n_2\pm\frac{1}{2},n_3\pm\frac{1}{2}}$ with suitably related eigenvalues M_{n_1,n_2,n_3} and $M_{n_1\pm\frac{1}{2},n_2\pm\frac{1}{2},n_3\pm\frac{1}{2}}$. These pairs of related harmonics were explicitly constructed in [9] and follow the schematic pattern given in fig. 1.

As already stressed fifteen years ago in [9], these relations are differential geometric identities on the compact 7–manifold X^7 that are required by consistency with the structure of UIR.s of the superconformal group $\mathrm{Osp}(\mathcal{N}|4)$. Because of

[2]The reader should be careful in comparisons with other papers and take into account that the definition of mass utilized in this paper is that of supergravity [27]. Specifically the mass squared of scalars is defined as the deviation from a conformal invariant equation, the mass of the gravitino is defined as the deviation from a Rarita Schwinger equation with supersymmetry, the mass squared of a spin one field is defined as the deviation from a gauge invariant equation.

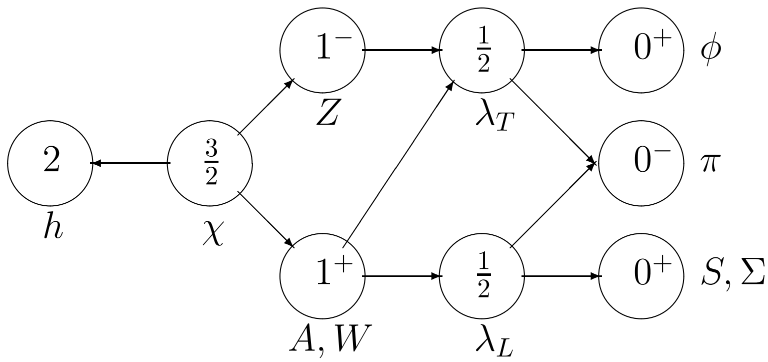

Figure 1: *"Internal"* supersymmetry relations between the harmonics of Kaluza Klein fields: for each pair of fields linked by an arrow the corresponding towers of harmonics can be related to each other by multiplications with a Killing spinor η^A.

this, it may at first sight appear that the universal mass relations analyzed in [9] do not contain further physical information besides the implications of supersymmetry. However, this is not the case, because of another aspect of eq.s (3.16 –3.26,3.28), whose consequence is precisely the existence of shadow multiplets.

The point is that the relations (3.16-3.26) between masses (or conformal weights) and eigenvalues of the internal Laplacian is quadratic rather than linear. Indeed the same harmonic always plays a double role since it appears in the expansion of two quite different Kaluza Klein fields. Combined with the supersymmetry relations produced by Killing spinors, this has the curious consequence that each supersymmetry multiplet of the Kaluza Klein spectrum is associated with another one made, so to say, by the second roots of the quadratic relations.

Examples We can illustrate the basic structure of this mechanism with two simple examples that will be quite relevant in our subsequent discussion.

Let us observe that the same scalar harmonic $Y^I(y)$ is associated both to the graviton field $h_{ab}(x)$, eq. (3.1) and to a scalar field $\Sigma^I(x)$, eq. (3.2). Using the mass relations (3.16 –3.26), we see that the shadow scalar of a "parent" graviton has mass

$$m_\Sigma^2 = m_h^2 + 176 + 24\sqrt{m_h^2 + 36} \tag{3.29}$$

In the case of the massless graviton, $m_h^2 = 0$, corresponding to the constant harmonic $Y = 1$, its shadow scalar Σ has (squared) mass

$$m_\Sigma^2 = 320 \ . \tag{3.30}$$

Using eq.s (3.28) this implies

$$E_\Sigma = 6 \ . \tag{3.31}$$

We conclude that in every $\mathrm{AdS}_4 \times X^7$ compactification there is always a *universal* scalar mode of conformal dimension $E = 6$ that is the shadow of the graviton. Its geometric origin is apparent from the second of equations (3.1): it is just the "breathing" mode corresponding to an overall dilatation of the internal manifold X^7. The interesting question is which supermultiplet this universal breathing mode belongs to in the case of supersymmetric compactifications. We will tackle this question in the next Section.

As it follows by inspection of eq.s (3.2), to the same vector harmonic Y^I_α we associate two vector fields: one, A_a, with mass given by eq. (3.22), the other, W_a, with mass given by eq. (3.21). The very initial idea of Kaluza Klein theory is that the isometries of the internal manifold give rise to massless gauge field in the compactified 4–dimensional theory. When we start from $D = 11$ M–theory there is a further aspect. Indeed to each Killing vector of the internal compact manifold we associate two rather than one vector fields. In addition to the KK massless gauge boson, we have its shadow massive vector. It also belongs to the adjoint representation of the isometry group, and it has fixed mass and dimension:

$$m^2_W = 192 \Rightarrow E_W = 5 \ . \tag{3.32}$$

Once again the natural question is which multiplet do these shadow vectors belong to in supersymmetric compactifications.

Before addressing this question, let us consider one more example involving fermionic fields. Comparing eq.s (3.6) and (3.7), we see that the same spinor harmonic Ξ^I appears both in the expansion of the gravitino $\psi_a(x,y)$ and as coefficient of the expansion of a longitudinal spin–1/2 field $\lambda_L(x)^I$. Using eq.s (3.24,3.26) the relation between the masses of the spin–3/2 and spin–1/2 modes pertaining to the same harmonic is:

$$m_\chi = -m_{\lambda_L} - 16 \ . \tag{3.33}$$

Applying eq. (3.33) to the case $m_{\lambda_L} = 0$, we see that each massless spin $\frac{1}{2}$ particle of this type generates a shadow massive gravitino with mass:

$$m_\chi = -16 \Rightarrow E_\chi = \frac{9}{2} \ . \tag{3.34}$$

Conversely, every massless gravitino produces a shadow massive spin–1/2 field with mass:

$$m_{\lambda_L} = -16 \Rightarrow E_{\lambda_L} = \frac{11}{2} \ . \tag{3.35}$$

3.2 The universal super-Higgs multiplet that shadows the massless $\mathcal{N} = 3$ graviton multiplet

We come now to answer the questions posed in the previous section by focusing on the very special case of $\mathcal{N} = 3$ compactifications. We want to show that, in such

$SD(2,3/2,0\|3)$			Kaluza Klein origin	
Spin	Energy	Isospin	Field	Harmonic
2	3	0	h_{ab}	$Y = 1$
$\frac{3}{2}$	$\frac{5}{2}$	1	χ^A_a	$\Xi^A = \eta^A$ (Killing spinor)
1	2	1	A^A_a	$Y^A_\alpha = \frac{1}{2}\,\epsilon^{ABC}\,\bar{\eta}^B\,\tau_\alpha\eta^C \equiv k^A_\alpha$
$\frac{1}{2}$	$\frac{3}{2}$	0	λ_L	$\Xi = \frac{1}{3}\,\epsilon^{ABC}\,\tau_\alpha\,\eta^A\,\bar{\eta}^B\,\tau^\alpha\eta^C \equiv \eta^0$

Table 2: The massless $\mathcal{N} = 3$ graviton multiplet and its Kaluza Klein origin.

cases, the shadow of the massless graviton multiplet is a long massive gravitino multiplet with a fixed energy, namely $E_0 = 3$.

The Osp(3|4) supermultiplets relevant to Kaluza Klein theory were analyzed in [29], and they were classified using the following nomenclature. By

$$SD(s_{\max}, E_0, J_0|3) \tag{3.36}$$

we denote an Osp(3|4) unitary irreducible representation (UIR) with maximal spin s_{max}, Clifford vacuum energy E_0 and $\mathrm{SU}(2)_R$ isospin J_0. The same notation can be extended to all Osp($\mathcal{N}$|4) representations; for example, we denote the $\mathcal{N} = 2$ UIRs by

$$SD(s_{\max}, E_0, y_0|2) \ , \tag{3.37}$$

where y_0 is the Clifford vacuum hypercharge.

Using the just introduced notation, Kaluza-Klein compactifications establish a very general shadowing relation between the following two representations that, algebraically, are totally unrelated:

$$SD(2, \tfrac{3}{2}, 0|3) \stackrel{\text{shadow}}{\longrightarrow} SD(\tfrac{3}{2}, 3, 0|3) \ . \tag{3.38}$$

The massless graviton multiplet The graviton multiplet $SD(2, \frac{3}{2}, 0|3)$ is *ultrashort* and corresponds to the gauge multiplet of $\mathcal{N} = 3, D = 4$ supergravity. Its algebraic structure, derived in [29], is displayed in the first three columns of Table 2. The last two columns of the same table are copied from Table VI of [9] and describe the Kaluza Klein origin of the multiplet.

As stressed in [19] and [9], we do not need to know the details of X^7 geometry. It suffices to know that it is a compact Einstein manifold admitting three Killing spinors *i.e.* three solutions η^A $(A = 1, 2, 3)$ of eq. (2.9): in terms of these we can construct all the harmonics of the graviton gauge multiplet. In particular we construct the Killing vectors $k^A_\alpha \equiv \frac{1}{2}\,\varepsilon^{ABC}\,\bar{\eta}^B\,\tau_\alpha\eta^C$ that generate the $\mathrm{SU}(2)_R$ factor in the isometry group $G = \mathrm{SU}(2)_R \times G'$ of X^7.

$SD(3/2, E_0 > 1, 0\|3)$ in AdS_4 space				Massive spin $\frac{3}{2}$ multiplet in Minkowski space		
Spin	Energy	Isospin	# of fields	Spin	SU(3) repr.	# of fields
$\frac{3}{2}$	$E_0 + \frac{3}{2}$	0	1	$\frac{3}{2}$	$\mathbf{1}$	1
1	$E_0 + 2$	1	6	1	$\mathbf{3}$	6
	$E_0 + 1$	1			$\mathbf{3}$	
$\frac{1}{2}$	$E_0 + \frac{5}{2}$	1	14	$\frac{1}{2}$	$\mathbf{3}$	14
	$E_0 + \frac{3}{2}$	$\begin{cases} 2 \\ 1 \end{cases}$			$\mathbf{8}$	
	$E_0 + \frac{1}{2}$	1			$\bar{\mathbf{3}}$	
0	$E_0 + 3$	0	14	0	$\mathbf{1}$	14
	$E_0 + 2$	$\begin{cases} 2 \\ 0 \end{cases}$			$\mathbf{6}$	
	$E_0 + 1$	$\begin{cases} 2 \\ 0 \end{cases}$			$\bar{\mathbf{6}}$	
	E_0	0			$\bar{\mathbf{1}}$	

Table 3: The Osp(3|4) admits a long gravitino representation $SD(\frac{3}{2}, E_0 > 1, 0|3)$ which has just the same structure as a massive spin 3/2 multiplet of Poincaré $\mathcal{N} = 3$ supersymmetry. Indeed this long representation has a smooth Poincaré limit, with arbitrary value of $E_0 > 1$ leading to an arbitrary mass of the gravitino.

The long gravitino multiplet Let us now go back to Osp(3|4) representation theory. According to the analysis of [29] for each energy label

$$E_0 > 1 \tag{3.39}$$

we have a long multiplet $SD(\frac{3}{2}, E_0, 0|3)$ whose general structure is displayed in table 3.

As shown in table 3, this multiplet has a smooth Poincaré limit and becomes the standard multiplet of a massive $\mathcal{N} = 3$ gravitino containing a total of 14 scalars, 14 spin–1/2 particles and 6 vector bosons. The relation between the isospin assignments in the anti de Sitter theory and the SU(3) assignments in the Poincaré theory is easily retrieved by recalling that the anti de Sitter R–symmetry group is just the maximal SO(3) subgroup of SU(3) defined as the set of 3×3 unitary matrices that are also real. Under this embedding, the SU(3) representations (which we label, as in table 3, by their dimensions) decompose as follows:

$$\begin{aligned} \mathbf{6}\,,\ \bar{\mathbf{6}} &\ \rightarrow\ J = 2 \oplus J = 1\ ; \\ \mathbf{3}\,,\ \bar{\mathbf{3}} &\ \rightarrow\ J = 1\ ; \\ \mathbf{1}\,,\ \bar{\mathbf{1}} &\ \rightarrow\ J = 0\ . \end{aligned} \tag{3.40}$$

This is sufficient to retrieve all the $\mathrm{SU}(2)_R$ representations appearing in the corresponding Osp(3|4) supermultiplet. Obviously the *mass eigenstates* $E = E_0 + 2$ and $E = E_0 + 1$ of isospin $J = 2$ can be linear combinations of the two $J = 2$ representations obtained from the branching of the SU(3) representations $\mathbf{6}$ and $\bar{\mathbf{6}}$ Similarly, the two mass eigenstates $E = E_0 + 3$ and $E = E_0$ of isospin $J = 0$ can be linear combinations of all the four $J = 0$ representations obtained from the branching of SU(3) representations. The coefficients of such linear combinations, just as the value of the parameter E_0 characterizing the entire multiplet, are dynamical quantities that depend on the detailed structure of the interaction Lagrangian in $D = 4$.

The super-Higgs phenomenon Indeed, in the context of four dimensional field theory, the only way to introduce a consistent coupling of a massive spin 3/2 particle is via supersymmetry breaking. One starts from a locally supersymmetric Lagrangian admitting a non–supersymmetric (or partially supersymmetric vacuum). The spectrum of the theory calculated in the broken phase around such a vacuum will contain a massive gravitino whose mass depends continuously on the *vev*.s of the scalar fields responsible for the super-Higgs mechanism. In particular an $\mathcal{N} = 3$ massive gravitino can emerge only from a partial supersymmetry breaking $\mathcal{N} = 4 \rightarrow \mathcal{N} = 3$. If this occurs in Minkowski space–time all the states in the multiplet have the same mass and a manifest SU(3) symmetry is preserved (right hand side of table 3); if the breaking occurs in AdS_4 space–time then the same states are organized in the pattern displayed on the left–hand side of table 3 ($SD(\frac{3}{2}, E_0, 0|3)$) and only $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ is preserved. Yet a local $\mathrm{SU}(3) \times U(1)$–symmetry is a necessary feature of the $\mathcal{N} = 3$ Lagrangian. In any case, in the $D = 4$ theory the value of E_0 depends on the *vev*.s of the scalar fields and is a continuous deformable parameter.

The "shadow" gravitino multiplet Our main point in this section is the explicit illustration of eq. (3.38). Indeed using the double role of the harmonics associated with the states of the massless $\mathcal{N} = 3$ graviton multiplet (table 2) and the universal relations derived in [9] we can construct an $\mathcal{N} = 3$ massive spin $\frac{3}{2}$ multiplet with predetermined energy (or conformal weight) $E_0 = 3$.

The suggested conclusion will be that any $\mathrm{AdS}_4 \times X^7$ compactification of M–theory with $\mathcal{N} = 3$ supersymmetry looks like a broken phase of an $\mathcal{N} = 4$ theory. We shall have a lot more to say about this conclusion in later sections but for the time being let us discuss how the shadow spin $\frac{3}{2}$ multiplet emerges in Kaluza Klein theory.

The effect is summarized in table 4 where we list the harmonics associated with each of the states of $SD(\frac{3}{2}, 3, 0|3)$.

All of the harmonics of $SD(\frac{3}{2}, 3, 0|3)$ can be expressed in a universal way (namely, independently from the details of the X^7 geometry); indeed they all can be expressed

Spin	E	$SU(2)_R$	Field	Mass	$M_{[\lambda_1,\lambda_2,\lambda_3]}$	Harmonic
$\frac{3}{2}$	$\frac{9}{2}$	$J=0$	χ^-	$m_\chi=-16$	$M_{\left(\frac{1}{2}\right)^2}=-16$	$\Xi=\eta^0$
1	5	$J=1$	W	$m_W^2=192$	$M_{1(0)^2}=48$	$Y_\alpha=k_\alpha^A$
	4	$J=1$	Z	$m_Z^2=96$	$M_{(1)^2 0}=96$	$Y_{[\alpha\beta]}=T^A_{[\alpha\beta]}$
$\frac{1}{2}$	$\frac{11}{2}$	$J=1$	λ_L	$m_{\lambda_L}=-16$	$M_{\left(\frac{1}{2}\right)^3}$	$\Xi=\eta^A$
	$\frac{9}{2}$	$J=2$ $J=0$	λ_T	$m_{\lambda_T}=12$	$M_{\frac{3}{2}\left(\frac{1}{2}\right)^2}=4$	$\Xi_\alpha=\widehat{\Omega}_\alpha^{(AB)}$ $\Xi_\alpha=\widehat{\Omega}_\alpha^{(A)}$
	$\frac{7}{2}$	$J=1$	λ_T	$m_{\lambda_T}=-8$	$M_{\frac{3}{2}\left(\frac{1}{2}\right)^2}=-16$	$\Xi_\alpha=\Theta_\alpha^A$
0	6	$J=0$	Σ	$m_\Sigma^2=320$	$M_{(0)^3}=0$	$Y=1$
	5	$J=2$ $J=0$	π	$m_\pi^2=192$	$M_{(1)^3}=-2$	$Y_{[\alpha\beta\gamma]}=\widehat{\Upsilon}^{(AB)}_{[\alpha\beta\gamma]}$ $Y_{[\alpha\beta\gamma]}=\widehat{\Upsilon}_{[\alpha\beta\gamma]}$
	4	$J=2$ $J=0$	ϕ	$m_\phi^2=96$	$M_{2(0)^2}=96$	$Y_{(\alpha\beta)}=H^m_{(\alpha\beta)}$ $Y_{(\alpha\beta)}=H^0_{(\alpha\beta)}$
	3	$J=0$	π	$m_\pi^2=32$	$M_{(1)^3}=3$	$Y_{[\alpha\beta\gamma]}=\mathcal{Q}_{\alpha\beta\gamma}$

Table 4: The harmonics of the universal long spin–3/2 multiplet appearing in Kaluza Klein compactifications with $\mathcal{N}=3$ supersymmetry. This multiplet suggests a super-Higgs mechanism $\mathcal{N}=4\to\mathcal{N}=3$

purely in terms of the Killing spinors. Let us write here such expressions, leaving, for the sake of clarity, the account of some derivations to Appendix B.

· *The massive gravitino harmonic* The top state of the multiplet is given by the massive gravitino of scale dimension $E=\frac{9}{2}$. This corresponds to an internal harmonic Ξ being an eigenstate of the operator (3.13) with eigenvalue $M_{(\frac{1}{2})^3}=-16$. Such a harmonic is

$$\Xi\equiv\eta^0=\tfrac{1}{3}\epsilon_{ABC}\,\tau_\alpha\eta^A\overline{\eta}^B\,\tau^\alpha\,\eta^C\ , \tag{3.41}$$

namely, by comparison with table 2, the harmonic of the massless dilatino sitting in the $\mathcal{N}=3$ graviton multiplet.

· *The harmonic of the massive W vector* The $\mathcal{N}=3$ massless graviton multiplet contains a triplet of massless graviphoton gauging the SU(2) R–symmetry group. Their harmonics are the SO(3) Killing vectors k_α^A admitting the following expression in terms of Killing spinors:

$$k_\alpha^A=\tfrac{1}{2}\,\epsilon^{ABC}\,\overline{\eta}^B\,\tau_\alpha\,\eta^C\ . \tag{3.42}$$

By means of the shadowing mechanism described in eq.s (3.21,3.32), the W–vectors associated with the harmonics (3.42) are the $s = 1, J = 1, E = 5$ states of the universal multiplet we are discussing.

· *The harmonic of the massive Z vector* The universal multiplet of table (3) contains a $J = 1$ triplet of massive vector fields with scale dimension $E = 4$. These take origin from the Z–sector of the Kaluza Klein expansion, namely they are associated with 2–forms on the compact 7–manifold X^7. The corresponding harmonic, that is an eigenstate of the Hodge de Rham operator (3.10) with eigenvalue $M_{(1)^2 0} = 96$, is constructed in terms of Killing spinors as follows:

$$T^A_{\alpha\beta} \equiv 7\,\overline{\eta}^A\,\tau_{\alpha\beta}\,\eta^0 + \overline{\eta}^A\,\tau_{[\alpha}D_{\beta]}\eta^0 \ . \tag{3.43}$$

Eq. (3.43) is an immediate consequence of eq.s (4.72b),(4.73) and (4.74) of [9]. Such equations explain how to construct a transverse 2–form with eigenvalue

$$M_{(1)^2 0} = (M_{(\frac{1}{2})^3} + 4)\,(M_{(\frac{1}{2})^3} + 8) \tag{3.44}$$

from any spinor Ξ with eigenvalue $M_{(\frac{1}{2})^3}$. In the case $M_{(\frac{1}{2})^3} = -16$, eq.(3.44) yields $M_{(1)^2 0} = 96$. So via the universal relations of [9] also this state is in the shadow of the massless graviton multiplet and, as it is explicitly evident from combining equations (3.43) with eq. (3.41), its harmonic is expressed solely in terms of Killing spinors.

· *The harmonic of the massive $E = 11/2$ spinors* Next the universal multiplet of table 3 contains a $J = 1$ triplet of massive spinors with scale dimension $E = \frac{11}{2}$. These are just the shadows of the massless gravitinos, according to the mass relation discussed in eq.(3.35). Indeed their harmonic is just given by the triplet of Killing spinors

$$\Xi = \eta^A \ . \tag{3.45}$$

Reading table 3 from top to bottom the next item to discuss should be the spinors with $E = \frac{9}{2}$. To understand their structure, it is however more convenient to make a leap and go first to the pseudo–scalars with $E = 5$.

· *The harmonic of the massive π scalars with $E = 5$* We leave the discussion of this case to Appendix B. The harmonics turn out to be given simply by

$$\Upsilon^{(AB)}_{\alpha\beta\gamma} = \overline{\eta}^A\,\tau_{\alpha\beta\gamma}\,\eta^B \ . \tag{3.46}$$

Since the three index τ–matrix is symmetric, the above expression is symmetric in the $\mathrm{SO}(3)_R$ indices A, B. Decomposing into the traceless and trace parts we obtain the $J = 2$ and $J = 0$ states.

· *The harmonic of the massive $E = 9/2$ spinors* We leave the expression of these harmonics to Appendix B.

· *The harmonic of the massive $E = 7/2$ spinors* Leaving the discussion to Appendix B, we give here only the resulting expression of the harmonics in terms

of Killing spinors:

$$\Theta^A_\alpha \equiv \epsilon^{ABC} \left(\frac{3}{16} \tau_{\alpha\mu\nu\rho} \, \eta^B \, D_\mu \, T^C_{\nu\rho} + \frac{9}{2} \tau_\mu \, \eta^B \, T^C_{\alpha\mu} \right) \ , \tag{3.47}$$

the quantities $T^A_{\alpha\beta}$ having been defined in eq. (3.43).

· *The harmonic of the massive π scalars with $E = 3$* This pseudo-scalar particle corresponds to the Clifford vacuum of the entire super-Higgs multiplet and, for this reason it is probably the most important state in the whole multiplet. In section 3.3 we shall see its 7–dimensional geometric interpretation in the case of the specific compactification $X^7 = N^{0,1,0}$ and later we shall discuss its superconformal correspondent in terms of world–volume gauge fields. Here we want to stress its universal character as a *shadow* of the massless graviton multiplet. Hence we present the construction of its harmonic in terms of Killing spinors.

We refer to eq.s (4.40b-4.43) of [9] which, applied to our case, show how to construct a 3–form of eigenvalue $M_{(1)^3} = 3$ starting from the harmonic Θ^A_α of eigenvalue $M_{\frac{3}{2}(\frac{1}{2})^2} = -16$. Indeed the relevant mass relation (4.43) of [9] is:

$$M_{(1)^3} = -\frac{1}{4}(M_{\frac{3}{2}(\frac{1}{2})^2} + 4) \ . \tag{3.48}$$

Hence, following the prescriptions of [9], we consider the combination

$$\widehat{\Pi}^{AB}_{\alpha\beta\gamma} = \bar{\eta}^A \, \tau_{[\alpha\beta} \, \Theta^B_{\gamma]} + \frac{1}{7} \bar{\eta}^A \, \tau_{[\alpha} \, D_\beta \, \Theta^B_{\gamma]} \tag{3.49}$$

which, by construction, is an eigenstate of the operator (3.11) with eigenvalue $M_{(1)^3} = 3$. A priori this object might contain isospins $J = 2, J = 1$ and $J = 0$ parts. However, explicit computation in a τ–matrix basis shows that

$$\widehat{\Pi}^{AB}_{\alpha\beta\gamma} = \delta^{AB} \, \mathcal{Q}_{\alpha\beta\gamma} \ . \tag{3.50}$$

Hence, in perfect agreement with Osp(3|4) representation theory we just have a spin $J = 0$ state and nothing else.

· *The harmonic of the Lichnerowitz scalars with $E = 4$* The last harmonics to be discussed are those associated with the $E = 0, J = 2 \oplus J = 0$ scalars. From the Kaluza Klein viewpoint these states originate from traceless fluctuations of the internal 7–dimensional metric. Their harmonics are therefore eigenfunctions of the Lichnerowitz operator (3.12) with eigenvalue $M_{2(0)^2} = 96$. To derive their expression in terms of Killing spinors it suffices to recall the mass relations (4.18-4.25) of [9] that teach us how to construct an eigenstate of the Lichnerowitz operator with eigenvalue

$$M_{2(0)^2} = (M_{\frac{3}{2}(\frac{1}{2})^2} + 4) \, (M_{\frac{3}{2}(\frac{1}{2})^2} + 8) \tag{3.51}$$

starting from an eigenstate of the Rarita-Schwinger operator (3.14) pertaining to the eigenvalue $M_{\frac{3}{2}(\frac{1}{2})^2}$. In our case the fermionic harmonic to start from is clearly

Θ^A_α given in eq. (3.47), which has $M_{\frac{3}{2}(\frac{1}{2})^2} = -16$. Hence, following the prescriptions of [9], we consider the following combination:

$$K^{AB}_{(\alpha\beta)} = \bar{\eta}^A \, \tau_{(\alpha} \, \Theta^B_{\beta)} + \frac{1}{3} \, \bar{\eta}^A \, D_{(\alpha} \, \Theta^B_{\beta)} \ , \tag{3.52}$$

which by construction is an eigenstate of eigenvalue $M_{2(0)^2} = 96$. One can explicitly check that (3.52)is symmetric in the isospin indices AB, so that it decomposes into a $J = 2$ plus a $J = 0$ representation. In particular, the combination that is an isospin singlet is

$$\mathcal{H}^0_{(\alpha\beta)} = \frac{1}{288} \left(K^{1,1}_{(\alpha\beta)} + K^{2,2}_{(\alpha\beta)} + K^{3,3}_{(\alpha\beta)} \right) \ . \tag{3.53}$$

This concludes our proof that in Kaluza Klein supergravity when we compactify M–theory on $\mathrm{AdS}_4 \times X^7$ with $\mathcal{N} = 3$ Killing spinors there is always a universal massive spin–3/2 multiplet of scale dimension

$$E_0 = 3 \tag{3.54}$$

which is just the necessary shadow of the massless graviton multiplet.

3.3 Freund-Rubin compactification on $N^{0,1,0}$

The space $N^{0,1,0}$, a particular instance of the series of 7-dimensional coset spaces named $N^{p,q,r}$ in the classification of [30], is the only 7-dimensional coset that, when used as a compactification manifold for 11D supergravity, can preserve $\mathcal{N} = 3$ supersymmetry [31]. The complete KK spectrum of the $N^{0,1,0}$ compactification has been derived in [32], and its Osp(3|4) multiplet structure elucidated in [29].

In the perspective of the AdS/CFT correspondence, the dual of the $N^{0,1,0}$ compactification should be a $\mathcal{N} = 3$ conformal field theory in $2 + 1$ dimensions. This theory is the subject of a paper [3] which is somehow companion to the present one.

The space $N^{0,1,0}$ can be simply defined as the coset space

$$\frac{G}{H} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)} \ , \tag{3.55}$$

where, using the Gell-Mann matrices λ^α as su(3) generators, the quotient is taken w.r.t. the U(1) subgroup generated by λ^8. The isotropy group of $N^{0,1,0}$ is $\mathrm{SU}(3) \times \mathrm{SU}(2)$; the SU(2) factor is the normalizer of the $U(1)$ action and, explicitly, it is generated by $\lambda^{1,2,3}$.

Let $\Omega^A = (\Omega^\alpha, \Omega^8)$ be the Maurer-Cartan forms for SU(3), namely let $\Omega = \Omega^A \lambda^A = g^{-1} dg$, with $g \in \mathrm{SU}(3)$, so that $d\Omega + \Omega \wedge \Omega = 0$. The vielbein corresponding to a generic $\mathrm{SU}(3) \times \mathrm{SU}(2)$-invariant metric are obtained from the coset vielbein Ω^α ($\alpha = 1, \ldots 7$) by rescaling independently the two groups associated to $\lambda^{\dot\alpha}$ ($\dot\alpha =$

$1,2,3)$ and $\lambda^{\widetilde{\alpha}}$ ($\widetilde{\alpha} = 4,5,6,7$). Indeed such a decomposition is respected both by the U(1) quotient and by the SU(2) action. Thus we have[3]:

$$B^{\alpha} = (\alpha^{-1}\Omega^{\dot{\alpha}}, \beta^{-1}\Omega^{\widetilde{\alpha}}) \ . \tag{3.56}$$

The spin connection $B^{\alpha\beta}$ and the curvature associated to these vielbein are straightforwardly computed (see [33]); in particular, the Ricci tensor is diagonal, with only two independent entries:

$$\begin{aligned} \mathcal{R}_{\dot{\alpha}\dot{\beta}} &= \frac{1}{4}\left(\alpha^2 + \frac{1}{2}\frac{\beta^4}{\alpha^2}\right)\eta_{\dot{\alpha}\dot{\beta}} \ ; & (3.57)\\ \mathcal{R}_{\widetilde{\alpha}\widetilde{\beta}} &= \frac{3}{4}\left(\beta^2 - \frac{1}{4}\frac{\beta^4}{\alpha^2}\right)\eta_{\widetilde{\alpha}\widetilde{\beta}} \ . & (3.58) \end{aligned}$$

Requiring the space to be Einstein, gives two possible values of the ratio β^2/α^2, namely $\beta^2/\alpha^2 = 2$ or $\beta^2/\alpha^2 = 2/5$. If we furthermore require that the Ricci tensor has exactly the value required by the Freund Rubin ansatz, eq. (2.7), we find four different possibilities, of which two preserve some supersymmetry.

The "standard" $N^{0,1,0}$ metric is obtained with the following rescalings:

$$\alpha = -4\ e \ , \quad \beta = \pm 4\sqrt{2}\ e \ . \tag{3.59}$$

It preserves $\mathcal{N} = 3$ supersymmetry. It is known [31] that, when $N^{0,1,0}$ is realized as the coset (3.55), its Killing spinors must actually be *constant*. With the rescalings (3.59), there are 3 independent constant spinors η^A ($A = 1,2,3$) that satisfy eq. (2.9), namely

$$-\frac{1}{4}B^{\alpha\beta}{}_{\gamma}\tau_{\alpha\beta}\,\eta^A = e\,\tau_\gamma\,\eta^A \ . \tag{3.60}$$

They transform as a *triplet* under the SU(2) part of the isometry, which therefore truly acquires the role of the R-symmetry group $\mathrm{SU}(2)_R$ for the 4-dimensional gauged supergravity that arises from the compactification.

There is a possible solution that differs from (3.59) only by the sign of the rescaling α. While the sign of β is irrelevant, because β appears quadratically also in the spin connection, reversing the sign of α amounts to reversing the sign of the spin connection (or, equivalently, to changing the orientation of the manifold). This solution with opposite orientation preserves no supersymmetry.

Of the two solutions with $\beta^2/\alpha^2 = 2/5$, the one with *opposite* orientation to the standard $N^{0,1,0}$ (3.59), that we name $\widetilde{N}^{0,1,0}$:

$$\alpha = \frac{20}{3}\ e \ , \quad \beta = \pm\frac{4}{3}\sqrt{10}\ e \ , \tag{3.61}$$

[3] Due to a different choice of structure constant, our rescaling α is minus twice the one used in [33].

preserves $\mathcal{N}=1$ supersymmetry. Indeed, one finds that the Killing spinor equation

$$-\frac{1}{4}\widetilde{B}^{\alpha\beta}{}_{\gamma}\tau_{\alpha\beta}\,\eta^0 = e\,\tau_\gamma\,\eta^0 \tag{3.62}$$

admits a single solution η^0, which is a *singlet* of $\mathrm{SU}(2)_R$.

The solution differing from the $\widetilde{N}^{0,1,0}$ solution (3.61) by the sign of α (so that it has the same orientation as the standard $N^{0,1,0}$) preserves no supersymmetry.

3.3.1 Harmonics of the shadow multiplet for $N^{0,1,0}$

Freund Rubin compactification on $N^{0,1,0}$, with rescalings (3.59), gives an $\mathcal{N}=3$ theory on AdS_4. As discussed in Section 3.2, in this compactification a "shadow" massive gravitino multiplet is generated. This suggests that truncating the theory to include the fields of this massive gravitino could be consistent, the resulting 4-dimensional theory corresponding to a *broken* $\mathcal{N}=4$ supergravity theory on AdS_4.

In section 3.2 all the harmonics corresponding to the fields sitting in the shadow multiplet of a generic $\mathcal{N}=3$ KK compactification have been expressed in terms of the $\mathrm{SU}(2)_R$ triplet of killing spinors η^A, whose existence is necessary for having residual $\mathcal{N}=3$ supersymmetry. The peculiarity of the $N^{0,1,0}$ compactification is that these spinors are actually *constant*. Thus all the harmonics of the shadow multiplet will in fact correspond to *constant* deformations of the internal space $N^{0,1,0}$. Tensor products of these harmonics will involve again only constant harmonics, without involving harmonics of other massive multiplets, indicating again the possibility of a consistent truncation.

Let us discuss here in detail the explicit form of those harmonics that will be relevant for the effective 4-dimensional theory we will construct, or whose geometrical interpretation can be somehow illuminating. We refer to table 4 for notations.

First of all, the harmonic of the massive gravitino, given in eq. (3.41), turns out to be exactly the η^0 of eq. (3.62), i.e. the constant spinor that would generate the single supersymmetry of the $\widetilde{N}^{0,1,0}$ compactification.

Killing vectors The Killing vectors k^A of the $\mathrm{SU}(2)_R$ isometry represent, according to the discussion in section 3.2, the harmonic associated to the massive vector W, of energy $E=5$. From their universal expression in terms of the Killing spinors η^A, eq. (3.42), one gets, in our conventions, the following explicit components:

$$k^1_\alpha = \delta_{\alpha,2}\ ,\qquad k^2_\alpha = -\delta_{\alpha,1}\ ,\qquad k^3_\alpha = \delta_{\alpha,3}\ . \tag{3.63}$$

The vectors $k^A = k^A_\alpha\partial_\alpha$ close the $\mathrm{SU}(2)_R$ isometry algebra, and the $\mathrm{SU}(2)_R$ transformation of the vielbein B^α is obtained via the Lie derivative:

$$\delta_A\, B^\alpha = l_{k^A}B^\alpha = k^A_\gamma(B^\alpha_{\beta|\gamma} - B^\alpha_{\gamma|\beta})\,B^\beta \equiv (J^A)^\alpha{}_\beta\,B^\beta\ , \tag{3.64}$$

where the Lie derivative is obviously simplified by the fact that the Killing vectors are constant in the vielbein basis. The action of $\mathrm{SU}(2)_R$ on the vielbein B^α, encoded

in the matrices $(J^A)^\alpha{}_\beta$, is reducible, and splits into a $J = 1$ representation (spanned by the first three vielbeins $B^{\dot\alpha}$) and a complex $J = 1/2$ representation spanned by the last four ones, $B^{\tilde\alpha}$.

Deformations of the internal metric The constant harmonic $Y = 1$ corresponds to a dilatation of the internal metric, namely to a common rescaling of all the vielbeins. This mode is associated to a four dimensional scalar field of energy $E = 6$, named $\Sigma(x)$ according to the notation of eq.(3.1) and Table 4.

The harmonics $H^0_{(\alpha\beta)}$ and $H^{(AB)}_{(\alpha\beta)}$ that give rise to scalar fields of energy $E = 4$, parametrize traceless deformations of the 7-dimensional metric, see the 3rd of eq.s (3.1). They are eigenfunctions of the Lichnerowitz operator (3.12) with eigenvalue $M_{2(0)^2} = 96$. They admit a universal expression in terms of Killing spinors that was derived in section 3.2 (see eq.s (3.52) and (3.53)).

It is interesting to retrieve the geometric interpretation of these tensors on the manifold $N^{0,1,0}$. To this effect we can consider the Lichnerowitz operator (3.12) applied to the space of constant, H-invariant, traceless symmetric 2-tensors. Via explicit calculations this space is found to be 9-dimensional.

Six states correspond to the eigenvalue $M_{2(0)^2} = 96$. One of these states is a singlet ($J = 0$) of $\mathrm{SU}(2)_R$, while the others correspond to $J = 2$. The explicit expression of the $J = 0$ eigenstate is

$$H^0_{\alpha\beta} = (\frac{4}{3}\,\delta_{\dot\alpha\dot\beta}, -\delta_{\tilde\alpha\tilde\beta}) \ . \tag{3.65}$$

It corresponds to a rescaling of the 7-dimensional vielbein that preserves the volume of $N^{0,1,0}$: it is a "squashing" deformation. We will name $\phi(x)$ the corresponding scalar fields of energy $E = 4$. The $J = 2$ eigenstates are non-diagonal, and correspond thus to deformations of the $N^{0,1,0}$ metric which are non-diagonal in the vielbein basis.

Three states correspond to an eigenvalue $M_{2(0)^2} = 0$. They are organized in a triplet ($J = 1$) of $\mathrm{SU}(2)_R$. The corresponding massless scalars belong to the Betti vector multiplet, namely to the additional massless vector multiplet that originates from the fact that $N^{0,1,0}$ admits a harmonic 2–form [9, 2].

If we allow both dilatation and squashing deformations, we see that we can in fact rescale independently the first three vielbeins, $V^{\dot\alpha}$, and the last four ones, $V^{\tilde\alpha}$. As we already noticed, see eq. (3.56), this is the most general type of rescaling that still gives a $\mathrm{SU}(2)_R$-invariant metric. This what we expected, since we considered only the deformations which are $\mathrm{SU}(2)_R$ singlets.

Summarizing, if we include in the effective theory the $\mathrm{SU}(2)_R$-invariant scalar fields Σ and ϕ, of $E = 6$ and $E = 4$, belonging to the "shadow" gravitino multiplet, we are considering, according to eq. (3.1), the following fluctuation of the internal metric:

$$h_{\alpha\beta}(x,y) \;\; = \;\; -12\Sigma(x)\,\delta_{\alpha\beta} + \phi(x)\,H^0_{\alpha\beta} \tag{3.66}$$

$$= \left((-12\,\Sigma(x) + \frac{4}{3}\,\phi)\,\delta_{\dot\alpha\dot\beta}, (-12\,\Sigma(x) - \,\phi)\,\delta_{\widetilde\alpha\widetilde\beta} \right) \ .$$

This corresponds[4] to a rescaling of the 7-dimensional vielbeins:

$$\begin{aligned} B^{\dot\alpha}(x,y) &= \mathrm{e}^{-6\Sigma(x)+\frac{2}{3}\phi(x)}\,\bar B^{\dot\alpha}(y) && (3.68)\\ B^{\widetilde\alpha}(x,y) &= \mathrm{e}^{-6\Sigma(x)-\frac{1}{2}\phi(x)}\,\bar B^{\widetilde\alpha}(y)\ , && (3.69)\end{aligned}$$

where on the r.h.s. the vielbein $\bar B^{\alpha}(y)$ are those of the standard $\mathcal{N} = 3$ $N^{0,1,0}$ solution. It is convenient for the discussion of the effective action to introduce the following combinations:

$$\begin{aligned} w(x) &= -18\Sigma(x) - 2\phi(x)\ ,\\ z(x) &= -18\Sigma(x) + \frac{1}{3}\phi(x)\ , \end{aligned} \tag{3.70}$$

in terms of which the rescalings of the vielbein read

$$\begin{aligned} B^{\dot\alpha}(x,y) &= \mathrm{e}^{\frac{1}{3}w(x)}\,\bar B^{\dot\alpha}(y) && (3.71)\\ B^{\widetilde\alpha}(x,y) &= \mathrm{e}^{-\frac{1}{6}w(x)+\frac{1}{2}z(x)}\,\bar B^{\widetilde\alpha}(y)\ . && (3.72)\end{aligned}$$

Turning on an internal 3-form The harmonics $\widehat\Upsilon_{[\alpha\beta\gamma]}$ and $\mathcal{Q}_{[\alpha\beta\gamma]}$ both corresponds to switching on, a the three-form field $A_{\alpha\beta\gamma}$ on the internal 7-dimensional space. They are eigentensors of the ${}^*d$ operator on three-forms, eq. (3.11), with eigenvalue $M_{(1)^3} = -2$, resp. $M_{(1)^3} = 3$, and are $\mathrm{SU}(2)_R$ singlets. They give rise to scalar fields which we name $\pi_1(x)$, resp. $\pi_2(x)$, of energy $E = 5$, resp. $E = 3$. In terms of Killing spinors, we have $\widehat\Upsilon_{[\alpha\beta\gamma]} = \bar\eta^A\tau_{\alpha\beta\gamma}\eta^A$, see eq. (3.46), while the expression of $\mathcal{Q}_{[\alpha\beta\gamma]}$ is given through eq.s (3.50,3.49,3.47). For $N^{0,1,0}$, these harmonics are constant tensors. They are found to be given by linear combinations of the two $\mathrm{SU}(2)_R$ invariants of order 3 one can construct with the vielbeins, using the Killing vectors (3.63) and the matrices J^A describing the $\mathrm{SU}(2)_R$ action (3.64):

$$\begin{aligned} \mathcal{Q}_{\alpha\beta\gamma} &= 8\,k^A_\alpha\,k^B_\beta\,k^C_\gamma\,\epsilon_{ABC} - 6\,k^A_{[\alpha}\,J^A_{\beta\gamma]}\ , && (3.73)\\ \widehat\Upsilon_{\alpha\beta\gamma} &= -3\,k^A_\alpha\,k^B_\beta\,k^C_\gamma\,\epsilon_{ABC} - 6\,k^A_{[\alpha}\,J^A_{\beta\gamma]}\ . && (3.74)\end{aligned}$$

Explicitly, they have the following non-zero components:

$$\widehat\Upsilon_{[123]} = 3\ ,\quad \widehat\Upsilon_{[147]} = \widehat\Upsilon_{[165]} = \widehat\Upsilon_{[246]} = \widehat\Upsilon_{[257]} = \widehat\Upsilon_{[345]} = \widehat\Upsilon_{[376]} = 1 \tag{3.75}$$

and

$$\mathcal{Q}_{[123]} = 2\ ,\quad \mathcal{Q}_{[147]} = \mathcal{Q}_{[165]} = \mathcal{Q}_{[246]} = \mathcal{Q}_{[257]} = \mathcal{Q}_{[345]} = \mathcal{Q}_{[376]} = -1\ . \tag{3.76}$$

[4]Indeed, from eq. (3.68) it follows

$$\delta_{\alpha\beta}V^\alpha(x,y)V^\alpha(x,y) = \left((\delta_{\alpha\beta} + h_{\alpha\beta}(x,y) + O(h^2)\right)\bar V^\alpha(y)\bar V^\beta(y)\ . \tag{3.67}$$

An alternative basis for the deformations parametrized by $\widehat{\Upsilon}$ and $\mathcal{Q}$, which will prove more convenient for later calculations, is provided by the tensors $\mathcal{S} = (\widehat{\Upsilon} + \mathcal{Q})/5$ and $\mathcal{T} = (2\widehat{\Upsilon} - 3\mathcal{Q})/5$, whose only non-vanishing components are

$$\mathcal{S}_{[123]} = 1 \ , \quad \mathcal{T}_{[147]} = \mathcal{T}_{[165]} = \mathcal{T}_{[246]} = \mathcal{T}_{[257]} = \mathcal{T}_{[345]} = \mathcal{T}_{[376]} = 1 \ . \tag{3.77}$$

Namely, the only components are of type $\mathcal{S}_{\dot\alpha\dot\beta\widetilde\gamma}$ and $\mathcal{T}_{\dot\alpha\widetilde\beta\widetilde\gamma}$.

Summarizing, to include in the effective theory the $\mathrm{SU}(2)_R$-invariant scalar fields $E = 5$ and $E = 3$, belonging to the "shadow" gravitino multiplet, we are considering, according to eq. (3.1), a fluctuation[5] of the 3-form field in the 7-dimensional directions:

$$A_{\alpha\beta\gamma}(x,y) = \pi_1(x)\, \widehat{\Upsilon}_{\alpha\beta\gamma} + \pi_2(x)\, \mathcal{Q}_{\alpha\beta\gamma} \ . \tag{3.78}$$

A parametrization that is more convenient for deriving the effective action is in terms of two different scalar fields $f(x)$ and $g(x)$, associated to the two subset of components of the three-form A that rescale under (3.68) in a fixed way:

$$\begin{aligned} A_{\dot\alpha\dot\beta\dot\gamma}(x,y) &= f(x)\, \mathrm{e}^{18\Sigma(x)-2\phi(x)}\, \mathcal{S}_{\dot\alpha\dot\beta\dot\gamma} = f(x)\, \mathrm{e}^{w(x)}\, \mathcal{S}_{\dot\alpha\dot\beta\dot\gamma} \ , \\ A_{\dot\alpha\widetilde\beta\widetilde\gamma}(x,y) &= g(x)\, \mathrm{e}^{18\Sigma(x)+\frac{1}{3}\phi(x)}\, \mathcal{T}_{\dot\alpha\widetilde\beta\widetilde\gamma} = g(x)\, \mathrm{e}^{z(x)}\, \mathcal{T}_{\dot\alpha\widetilde\beta\widetilde\gamma} \ , \end{aligned} \tag{3.79}$$

where we have explicitly taken into account the effect of the rescaling (see eq. (3.68)) so that the tensors $\mathcal{S}_{\dot\alpha\dot\beta\dot\gamma}$ and $\mathcal{T}_{\dot\alpha\widetilde\beta\widetilde\gamma}$ in the r.h.s. remain the constant ones of eq. (3.77). Eq. (3.79) also displays the geometrical meaning of the rescalings w and z. Of course, f and g will no longer be mass eigenstates of the effective theory. Their relations to the mass eigenstates π_1, π_2 follows from the relations among $\mathcal{S}, \mathcal{T}$ and $\widehat{\Upsilon}, \mathcal{Q}$.

3.3.2 The shadow of the massless vector multiplets

As we observed in section 3.1, see eq. (3.32), every massless vector that appears in Kaluza–Klein theory has a massive vector companion, named W, of mass $m_W^2 = 192$, in the adjoint representation of the same gauge group.

We have seen what is the role of the massive shadows of the graviphotons in an $\mathcal{N} = 3$ compactifications. They are part of a universal massive spin–3/2 multiplet that hints to a hidden super-Higgs mechanism. However, in addition to the graviphotons gauging the R–symmetry, we also have the massless vectors that belong to massless vector multiplets and correspond to *flavor* symmetries of the dual conformal field theory. In which kind of multiplets do these "shadows" of the flavor currents fit?

The answer to this question cannot be given in universal terms since flavor symmetries depend on the specific choice of the X^7 manifold. Yet it can be inferred

[5]Notice that the field $A_{\alpha\beta\gamma}$ coincides with its fluctuation $a_{\alpha\beta\gamma}$ appearing in eq. (3.1) as the background value of $A_{\alpha\beta\gamma}$ is zero.

$SD(2,5/2,1\|3)$ (in the **8** of $SU(3)_{\text{flavor}}$)		
Spin	Energy	Isospin
2	4	1
$\frac{3}{2}$	$\frac{9}{2}$	$1\oplus 0$
	$\frac{7}{2}$	$2\oplus 1\oplus 0$
1	5	0
	6	$2\oplus 1\oplus 1\oplus 0$
	3	$2\oplus 1\oplus 0$
$\frac{1}{2}$	$\frac{9}{2}$	1
	$\frac{7}{2}$	$2\oplus 1\oplus 1\oplus 0$
	$\frac{5}{2}$	1
0	4	1
	3	$1\oplus 0$

Table 5: In $N^{0,1,0}$ compactification, the shadow of the massless vector multiplet in the adjoint of $SU(3)_{\text{flavor}}$ is the exceptional short $\mathcal{N}=3$ graviton multiplet $SD(2,\frac{5}{2},1|3)$ with $E_0=\frac{5}{2}$ and $J_0=1$.

that the *type* of multiplet to which such shadows belong is a universal feature of Kaluza Klein compactifications.

In the case of the manifold $N^{0,1,0}$, we can read off the desired answer from the complete spectrum of $\mathrm{Osp}(3|4)\times SU(3)$ supermultiplets derived in [29, 32]. In particular, looking at eq. (8) of that paper and at table 5, we can consider the case $k=1, j=0$. Correspondingly, we find a long graviton multiplet $SD(2,E_0=\frac{5}{2},J_0=1|3)$ in the adjoint representation **8** of the flavor SU(3) group. Indeed, recalling eq. (7) of [29], for $M_1=M_2=1$ and $J_0=1$ we have $H_0=64$, hence $E_0=5/2$. This long rational multiplet is displayed in table 5 and includes both the W-shadows of the massless SU(3) gauge bosons and the massive gravitinos that are shadows of the massless SU(3) gauginos. Since it goes up to spin $s_{\max}=2$ this long multiplet cannot be used to describe any Higgs or super-Higgs mechanism. This is in agreement with another feature that also points in the same direction. Indeed, differently from the harmonics of the universal super-Higgs multiplet, the harmonics of this long multiplet are not *constant*. So we can expect that tensor products thereof can contains the harmonics of other massive multiplets. This excludes that the truncation of Kaluza Klein theory to massless multiplets plus this particular long multiplet can be a consistent one.

Spin	Energy	Hypercharge	Mass (^{2})	K.K. origin
1	5	0	192	W
$\frac{1}{2}$	$\frac{11}{2}$	-1	-16	λ_L
$\frac{1}{2}$	$\frac{11}{2}$	1	-16	λ_L
$\frac{1}{2}$	$\frac{9}{2}$	-1	12	λ_T
$\frac{1}{2}$	$\frac{9}{2}$	1	12	λ_T
0	6	0	320	Σ
0	5	-2	192	π
0	5	2	192	π
0	5	0	192	π
0	4	0	96	ϕ

Table 6: The universal long vector multiplet that in $\mathcal{N}=2$ compactifications is the shadow of the massless graviton multiplet

3.4 Comparison with $\mathcal{N}=2$ compactifications

The same shadowing mechanism may be directly tested also in the case of $\mathcal{N}=2$ compactifications using the results on Kaluza Klein spectra obtained in [34, 2, 4]. For instance, looking at the spectrum of $M^{1,1,1}$ given in [34], one finds the existence of a long vector multiplet with the structure displayed in table 6. Indeed, it suffices to look at eq. (3.20) of [34], and set $M_1 = M_2 = J = 0$ to realize that the corresponding long vector multiplet shown in table 3 of the same paper has $E_0 = 4$, $y_0 = 0$ and exactly reproduces table 6 of the present paper. On the other hand, comparison of table 6 with table 3 reveals that the field components of this long vector multiplet are just a subset of the universal gravitino shadow multiplet of $\mathcal{N}=3$ compactifications. In particular, we observe the presence of the $E=6$ scalar mode Σ and of the massive W vector with $E=5$ that is the shadow partner of the SO(2) R-symmetry gauge field. Therefore we can easily conclude that table 6 displays the universal structure of the shadow multiplet of the massless supergravity multiplet in $\mathcal{N}=2$ compactifications. The difference is that in this case this universal multiplet reaches only spin 1 and not spin 3/2. Hence there is no room for superHiggs interpretation in this class of compactifications.

4 Partial breaking from $\mathcal{N}=4$ to $\mathcal{N}=3$ and the coset structure of supergravity

As we argued in the previous sections, in an $\mathcal{N}=3$ compactification on $\mathrm{AdS}_4 \times X^7$, the truncation of Kaluza Klein theory to the massless modes plus the massive gravitino multiplet should be consistent. The corresponding 4–dimensional Lagrangian

should then describe a spontaneously broken phase of $\mathcal{N}=4$ supergravity. Hence we turn our attention to the partial supersymmetry breaking mechanism from $\mathcal{N}=4$ to $\mathcal{N}=3$ in the context of 4–dimensional field theory. Here we find an intriguing surprise. Using the standard formulation of the $\mathcal{N}=4$ theory, which so far has been considered unique, it turns out that in an AdS$_4$-vacuum with $\mathcal{N}=3$ residual unbroken supersymmetry there is an upper bound on the scaling dimension (or mass) of the broken gravitino multiplet:

$$E_0\left(\langle\mu\rangle\right) < 3 \ , \tag{4.1}$$

where $\langle\mu\rangle$ stands for the vacuum expectation values (v.e.vs) of the scalar fields μ, that we name "moduli", in the theory.

Actually, as we are going to explain, if we require $\mathcal{N}=4 \to \mathcal{N}=3$ partial breaking, the true moduli space $\mathcal{M}^{\mathrm{p.b.}}$ of such vacua is given by a small subset of scalar fields that are singlet under the residual $\mathrm{SO}(3)_R$ R–symmetry group and under all unbroken gauge symmetries. These moduli fill two copies of the upper complex plane equipped with the Poincaré metric:

$$\mathcal{M}^{\mathrm{p.b.}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)} \ . \tag{4.2}$$

The coordinates of $\mathcal{M}^{\mathrm{p.b.}}$ are suitable combinations of the four isospin singlets $(J=0)$ with energies E_0+3, E_0+2, E_0+1 and E_0 appearing in table 3. From the point of view of Kaluza-Klein theory, these coordinates parametrize the corresponding metric and internal photon deformations we have already discussed. In the context of 4–dimensional $\mathcal{N}=4$ supergravity, the two factors of $\mathcal{M}^{\mathrm{p.b.}}$ have different origins. The manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$ contains the scalar degrees of freedom, μ_{grav}, of the $\mathcal{N}=4$ graviton multiplet, while $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ is parametrized[6] by a subset, μ_{vec}, of the $6\times n$ scalars belonging to n vector multiplets coupled to supergravity.

The main issue of the present section is to show that, in the context of standard $\mathcal{N}=4$ supergravity, the scaling dimension E_0 is actually a function only of the vector multiplet moduli:

$$E_0 = E_0\left(\langle\mu_{\mathrm{vec}}\rangle\right) \ , \tag{4.3}$$

and that it satisfies the bound in eq.(4.1). Moreover, the bound is saturated only at the boundary of the "matter" moduli space $\mathrm{SO}(2,1)/\mathrm{O}(2)$. This means that, in the appropriate Poincaré metric, the vacuum realized by Kaluza Klein theory, which is perfectly regular, is at *infinite distance* from all other vacua of standard $\mathcal{N}=4$ supergravity. This suggests the existence of a more general formulation of $\mathcal{N}=4$ supergravity able to accommodate also the vacuum that is explicitly realized by

[6]Clearly, $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ is locally equivalent to $\mathrm{SU}(1,1)/\mathrm{U}(1)$ (or to $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$; however we will see later that, at the level of the embedding into the full moduli space, the correct description is the first one.

Kaluza Klein theory. Since unitarity imposes only the bound:

$$E_0 > 1 \; , \tag{4.4}$$

it is strongly suggested that the bound (4.1) simply divides the full supergravity moduli space of in two or more open charts. The first chart, which can be described by the standard Lagrangian, contains all the vacua for which the massive gravitino corresponds to a unitary representation with $E_0 < 3$. The other open charts, that should be covered by one or more "shadow" Lagrangians, should contain all the unitary representations with $E_0 > E_{\rm min}$, for some $E_{\rm min} < 3$.

In section 5 we argue about the possible existence of similar shadow extensions for all $\mathcal{N} \geq 4$ supergravities.

4.1 General aspects of partial supersymmetry breaking

To discuss the partial breaking $\mathcal{N} = 4 \rightarrow \mathcal{N} = 3$ we begin by recalling some very general aspects of the super-Higgs mechanism in extended supergravity that were codified in the literature of the early and middle eighties [35, 13, 36, 37] (for a review see chapter II.8 of [27]) and were further analyzed and extended in the middle nineties [38, 39, 40, 41].

A vacuum of supergravity is simply identified by a configuration of constant v.e.v.s of the scalar fields, $\langle\phi\rangle^i = \phi^i_0$, that is an extremum of the scalar potential:

$$\left.\frac{\partial V}{\partial \phi^i}\right|_{\phi=\phi_0} = 0 \; , \tag{4.5}$$

and a metric[7] $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ that is either that of Minkowski space, if $V(\phi_0) = 0$, or that of anti de Sitter space AdS_4, if $V(\phi_0) < 0$, or that of de Sitter space dS_4, if $V(\phi_0) > 0$. Therefore, in relation with the super-Higgs mechanism, there are just three relevant items of the entire supergravity construction that we have to consider.

1. The *gravitino mass matrix* $S_{AB}(\phi)$, namely the non-derivative scalar field dependent term that appears in the gravitino supersymmetry transformation rule:

   $$\delta\psi_{A|\mu} = \text{derivative terms} + S_{AB}\,(\phi)\;\gamma_\mu\,\epsilon^B \; , \tag{4.6}$$

   and reappears as a mass term in the Lagrangian:

   $$\mathcal{L}^{\rm SUGRA} \;=\; \ldots + \text{const}\,\left(S_{AB}(\phi)\,\psi^A_\mu\,\gamma^{\mu\nu}\,\psi^B_\nu \;+\; S^{AB}(\phi)\,\psi_{A|\mu}\,\gamma^{\mu\nu}\,\psi_{B|\nu}\right) \tag{4.7}$$

[7] Here we temporarily use the curved 4D space-time indices $\mu, \nu, \ldots$ in place of the "flat" ones $a, b, \ldots$ to avoid possible conflicts of notations.

2. The *fermion shifts*, namely the non-derivative scalar field dependent terms in the supersymmetry transformation rule of the spin $\frac{1}{2}$ fields :

$$\begin{aligned} \delta\,\lambda^i_R &= \text{derivative terms} + \Sigma_A{}^i\,(\phi)\,\epsilon^A\ , \\ \delta\,\lambda^i_L &= \text{derivative terms} + \Sigma^{A|i}\,(\phi)\,\epsilon_A\ . \end{aligned} \tag{4.8}$$

3. The scalar potential itself, $V(\phi)$.

These three items are related by a general supersymmetry Ward identity, firstly discovered in the context of gauged $\mathcal{N}=8$ supergravity [42] and later extended to all supergravities [35, 36, 37], that, in the conventions of [13, 14, 46, 52] reads as follows:

$$3\,S_{AC}\,S^{CB} - \frac{1}{2}K_{i,j}\Sigma_A{}^i\Sigma^{B|j} = -\delta^B_A\,V\ , \tag{4.9}$$

where $K_{i,j}$ is the kinetic matrix of the spin–1/2 fermions. The numerical coefficients appearing in (4.9) depend on the normalization of the kinetic terms of the fermions, while $A,B,\ldots=1,\ldots,\mathcal{N}$ are SU($\mathcal{N}$) indices that enumerate the supersymmetry charges. We also follow the standard convention that the upper or lower position of such indices denotes definite chiral projections of Majorana spinors, right or left depending on the species of fermions considered[8]. The position denotes also the way of transforming of the fermion with respect to SU($\mathcal{N}$), with lower indices in the fundamental and upper indices in the fundamental bar. In this way we have $S^{AB}=(S_{AB})^\star$ and $\Sigma_A{}^i=\left(\Sigma^{B|i}\right)^\star$. Finally, the index i is a collective index that enumerates all spin–1/2 fermions present in the theory.

In the case of $\mathcal{N}=4$ supergravity the spin–1/2 fermions are the dilatino and the gauginos. The dilatino χ^A belongs to the graviton multiplet, and has a left-chiral projection in the $\bar{\mathbf{4}}$ of SU(4), implying a fermion shift:

$$\delta\,\chi^A = \text{derivative terms} + \Sigma_{AB}\,(\phi)\,\epsilon^B\ . \tag{4.10}$$

The gauginos λ^I_A belong to the vector multiplets and have a left-chiral projection in the $\mathbf{4}$ of SU(4); the index I belongs to the adjoint of the gauge group. Their fermion shifts are

$$\delta\,\lambda^I_A = \text{derivative terms} + \Sigma_A{}^{B|I}\,(\phi)\,\epsilon_B\ . \tag{4.11}$$

A vacuum configuration ϕ_0 that preserves $\mathcal{N}_0$ supersymmetries is characterized by the existence of $\mathcal{N}_0$ vectors $\rho^A_{(\ell)}$ ($\ell=1,\ldots,\mathcal{N}_0$) of SU($\mathcal{N}$), such that

$$\begin{aligned} S_{AB}\,(\phi_0)\,\rho^A_{(\ell)} &= \mathrm{e}^{\mathrm{i}\theta}\sqrt{\tfrac{-V(\phi_0)}{3}}\,\rho_{A(\ell)}\ , \\ \Sigma_A{}^i\,(\phi_0)\,\rho^A_{(\ell)} &= 0\ , \end{aligned} \tag{4.12}$$

[8]For instance, we have $\gamma_5\,\epsilon_A=\epsilon_A$ and $\gamma_5\,\epsilon^A=-\epsilon^A$.

where θ is an irrelevant phase. Indeed, consider the spinor

$$\epsilon^A(x) = \sum_{\ell=1}^{\mathcal{N}_0} \rho^A_{(\ell)} \epsilon^{(\ell)}(x) \ , \tag{4.13}$$

where $\epsilon^{(\ell)}(x)$ are $\mathcal{N}_0$ independent solutions of the equation (2.10) for covariantly constant spinors in AdS$_4$ (or Minkowski space) with $4\,e = \sqrt{-V(\phi_0)/3}$. Then it follows that under supersymmetry transformations of parameter (4.13) the chosen vacuum configuration $\phi = \phi_0$ is invariant[9]. That such a configuration is a true vacuum follows from another property proved, for instance, in [37]: all vacua that admit at least one vector ρ^A satisfying eq. (4.12) are automatically extrema of the potential, namely they satisfy eq. (4.5).

An important general feature of extended supergravities, $\mathcal{N} \geq 2$, is that the fermion shifts and the gravitino mass–matrix are uniquely determined by the *gauging* of the theory and are proportional to the gauge group coupling constants g_i. Indeed there are very general formulae for these objects expressing them in terms of geometrical data of the scalar manifold and of the structure constants of the gauge group (or of representation matrices if, in addition to vector multiplets, also hyper-multiplets are present). Hyper-multiplets are present only for the case $\mathcal{N} = 2$, whose most general form and gauging is discussed in [43] and whose partial breaking is discussed [40, 41, 39, 38].

For $\mathcal{N} \geq 5$ hyper and vector multiplets are absent and the scalar manifold is believed to be a uniquely fixed non–compact coset space. For $\mathcal{N} = 3, 4$ there are no hyper-multiplets and, in addition to the graviton multiplet, there are at most vector multiplets. Also in this case, the geometry of the scalar manifold is fixed to be that of a non–compact coset space

$$\mathcal{M}_{(n)} = \frac{G_{(n)}}{H_{(n)}} \ , \tag{4.14}$$

depending on the number n of vector multiplets. It is usually required that the subgroup $H_{(n)}$, the maximal compact subgroup of $G_{(n)}$ is of the form

$$H_{(n)} = \mathrm{SU}(\mathcal{N}) \times H'_{(n)} \ . \tag{4.15}$$

This is due to the assumption, always made in supergravity constructions [44, 45], that an $\mathcal{N}$–extended locally supersymmetric Lagrangian should have a *local* $\mathrm{U}(1) \times \mathrm{SU}(\mathcal{N})$ invariance, $\mathrm{U}(\mathcal{N}) \sim \mathrm{U}(1) \times \mathrm{SU}(\mathcal{N})$ being the automorphism group of the supersymmetry algebra with $\mathcal{N}$ supercharges[10]. This assumption, that leads to *unique* choices of the scalar manifolds $\mathcal{M}_{(n)}$ manifolds and, correspondingly, to

[9]As already stressed, the v.e.v.s of all the fermions are zero and equation (4.12) guarantees that they remain zero under supersymmetry transformations of parameters (4.13).

[10]A subtle distinction occurs at the level of the U(1)–factor that is missing only in the $\mathcal{N} = 8$ case, where the graviton multiplet is CPT self–adjoint.

unique supergravity Lagrangians, is motivated by the requirement that all supercharges should be treated on the same footing and that at least one vacuum with the full $\mathcal{N}$–extended supersymmetry should be present in the theory one wants to construct.

The point we want to make in the present paper, which is strongly suggested by the shadow mechanism we have discovered in Kaluza–Klein theory, is the following. The enforcement of $\mathrm{SU}(\mathcal{N})$ local symmetry may be an unnecessary and too strong constraint for $\mathcal{N}$–extended local supersymmetry. Breaking $\mathrm{SU}(\mathcal{N})$ treats the $\mathcal{N}$ supercharges on an unequal footing and prevents the existence of vacua preserving all of them. Yet if we are looking for Lagrangians where one or more supersymmetries are always broken in any choice of the vacuum, dismissing $\mathrm{SU}(\mathcal{N})$ may be the appropriate decision. Said in different words, what we are possibly looking for is not the construction of $\mathcal{N}$–extended supergravity, rather it is the coupling of $\mathcal{N} - \mathcal{N}_0$ *massive* gravitino multiplets to $\mathcal{N}_0$–extended supergravity. Since the only way to give mass to a gravitino is via a super-Higgs mechanism, the two programmes amount to the same thing. Yet, in the second perspective, the need for $\mathrm{SU}(\mathcal{N})$ local symmetry disappears. We should just be satisfied with $\mathrm{SU}(\mathcal{N}_0)$ symmetry. We shall argue that this is the necessary step in order to avoid the bound (4.1) implied by the standard SU(4)–symmetric $\mathcal{N} = 4$ theory.

4.2 Standard SU(4) symmetric $\mathcal{N} = 4$ supergravity

Let us for the moment step back to the standard formulation of $\mathcal{N} = 4$ supergravity and analyze its implications.

The $\mathcal{N} = 4$ graviton multiplet contains a scalar and a pseudo-scalar that can be combined into a complex field S parametrizing the coset manifold $\mathrm{SU}(1,1)/U(1)$. On the other hand, each $\mathcal{N} = 4$ vector multiplet contains 6 scalars. In the standard coupling of n vector multiplets these $6 \times n$ scalars are identified with the coordinates of the coset manifold $\mathrm{SO}(6,n)/(\mathrm{SO}(6) \times \mathrm{SO}(n))$. Hence in standard $\mathcal{N} = 4$ supergravity the choice of the coset manifold [11] (4.14) is as follows [49, 46, 50]:

$$\mathcal{M}^{(\mathcal{N}=4)}_{(n)} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,n)}{\mathrm{SO}(6) \times \mathrm{SO}(n)} \ . \tag{4.16}$$

The $\mathrm{U}(4) = \mathrm{U}(1) \times \mathrm{SU}(4)$ symmetry arises because of the U(1) denominator in the coset describing the graviton multiplet scalar and because of the local isomorphism $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ in the isotropy subgroup of the coset describing the matter multiplet scalars.

[11] The $\mathrm{SO}(6,n)$ symmetry has been derived in direct supergravity constructions using the conformal tensor calculus [46] but it has also a string theory origin. Compactifying the heterotic string on T^6 gives at the massless level $\mathcal{N} = 4$ supergravity coupled to $n = 22$ vector multiplets. The symmetry $\mathrm{SO}(6,22)$ arises [47, 48] as the symmetry of the lattice of momenta and winding numbers.

Let us notice, as it will be important later, that at the level of global supersymmetry there is no difference between $\mathcal{N}=4$ and $\mathcal{N}=3$ theories. The field content of the vector multiplets is the same, and we always have $6 \times n$ scalars. The difference arises only at the level of coupling to supergravity. As it was shown in [51], if we couple the same gauge theory to the $\mathcal{N}=3$, rather than to the $\mathcal{N}=4$, graviton multiplet the scalars do not parametrize the coset manifold $\mathrm{SO}(6,n)/(\mathrm{SO}(6)\times\mathrm{SO}(n))$ but have to be interpreted as the coordinates of the manifold

$$\mathcal{M}_{(n)}^{(\mathcal{N}=3)} = \frac{\mathrm{SU}(3,n)}{\mathrm{SU}(3)\times\mathrm{U}(1)\times\mathrm{SU}(n)} \ , \tag{4.17}$$

which replaces (4.16) and displays only a local $\mathrm{SU}(3)\times U(1)$ symmetry. $\mathcal{N}=3$ supergravity coupled to n vector multiplets both in un-gauged and gauged versions has been fully constructed in [51] using (4.17) as a starting point.

We might follow the perspective advocated some lines above and try to further couple to such a theory also a massive $\mathcal{N}=3$ spin–1/2 multiplet, whose field content and SU(3) representation assignments have been displayed in the right hand part of table 3. Should such a coupling necessarily involve an extension of the local symmetry from SU(3) to SU(4) and of the duality symmetries from $\mathrm{SU}(3,n)$ to $\mathrm{SO}(6,n)$? This is what happens if we think of such a coupling as a broken phase of "standard" $\mathcal{N}=4$ supergravity. In this section we want to argue that what enforces the bound (4.1) is precisely the $\mathrm{SO}(6,n)$ symmetry. This suggests the existence of a shadow supergravity where such a symmetry is dismissed.

Symplectic embeddings and the de Roo and Wagemans formulation Let us now introduce the formulae for the gravitino mass matrix (4.6) and for the fermion shifts (4.10,4.11) as they arise in standard $\mathcal{N}=4$ supergravity. For this we utilize the formulation by de Roo and Wagemans [13, 46, 52], since it is the most general one so far available. Indeed these authors have been the only ones to realize the presence of some additional phase parameters that had escaped notice in the other constructions ([49, 50] or chapter II.8 of [27]) and that turn out to be essential for the partial $\mathcal{N}=4\to\mathcal{N}=3$ breaking mechanism [14].

Consider a family of $\mathrm{SU}(1,1)$ matrices $\mathcal{S}(u_1,u_2)$ that parametrize the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$. Namely, set

$$\mathcal{S}(u_i) \equiv \begin{pmatrix} \varphi_1(u_i) & \varphi_2^*(u_i) \\ \varphi_2(u_i) & \varphi_1^*(u_i) \end{pmatrix} \ , \tag{4.18}$$

with the constraint $|\varphi_1|^2 - |\varphi_2|^2 = 1$. The choice of an explicit coset parametrizations is conceptually irrelevant for the construction of supergravity but there are special choices (or gauges) that can simplify calculations in a substantial way and that we shall discuss in a moment.

Similarly, let $L^{\Lambda}{}_{\Gamma}(\phi_i)$, function of $6n$ parameters ϕ_i, be a family of $\mathrm{SO}(6,n)$ matrices parametrizing the coset $\mathrm{SO}(6,n)/(\mathrm{SO}(6)\times\mathrm{SO}(n))$. Thus,

$$L^{\Lambda}{}_{\Gamma}\, L^{\Sigma}{}_{\Delta}\, \eta_{\Lambda\Sigma} = \eta_{\Gamma\Delta} \ , \tag{4.19}$$

where we use the Killing metric

$$\eta_{\Lambda\Sigma} = \mathrm{diag}(\underbrace{-,\ldots,-}_{6},\underbrace{+,\ldots,+}_{n}) \ . \tag{4.20}$$

Next we reassemble the coset elements L_a^Λ $(a = 1,\ldots 6)$ into

$$\Phi^\Lambda_{AB} = \tfrac{1}{2}\sum_{x=1}^{3}\left(L_x{}^\Lambda\, J^{(-)x}_{AB} + \mathrm{i}\, L_{x+3}{}^\Lambda\, J^{(+)x}_{AB}\right) \tag{4.21}$$

satisfying the reality condition

$$\Phi^{\Lambda|AB} \equiv \left(\Phi^\Lambda_{AB}\right)^\star = -\epsilon^{ABCD}\,\Phi^\Lambda_{CD} \ . \tag{4.22}$$

In eq. (4.21) we made use of the 't Hooft matrices $J^{(\pm)x}_{AB}$ $(x = 1,2,3)$ which provide a complete basis for 4×4 antisymmetric matrices. They are (anti)self-dual: $J^{(\pm)x}_{AB} = \pm\epsilon_{ABCD}J^{(\pm)x}_{AB}$; the $J^{(+)}$ commute with the $J^{(-)}$ and both realize the quaternion algebra.

We consider the gauging of a $6+n$–dimensional gauge group, in general composed of several simple factors. The gravitino mass-matrix and the fermion shifts determined by de Roo and Wagemans are then

$$\begin{aligned}
S_{AB} &= -\frac{2}{3}\sum_{i=1}^{k} g_i\,\left(e^{\mathrm{i}\theta_i}\varphi_1^\star + e^{-\mathrm{i}\theta_i}\varphi_2^\star\right)\,\Phi^{\Lambda_i}_{AC}\,\Phi^{CD|\Delta_i}\,\Phi^{\Sigma_i}_{DB}\,f_{\Lambda_i\Sigma_i}{}^{\Omega_i}\,\eta_{\Delta_i\Omega_i} \\
\Sigma_{AB} &= -\frac{4}{3}\sum_{i=1}^{k} g_i\,\left(e^{-\mathrm{i}\theta_i}\varphi_1 - e^{\mathrm{i}\theta_i}\varphi_2\right)\,\Phi^{\Lambda_i}_{AC}\,\Phi^{CD|\Delta_i}\,\Phi^{\Sigma_i}_{DB}\,f_{\Lambda_i\Sigma_i}{}^{\Omega_i}\,\eta_{\Delta_i\Omega_i} \\
\Sigma_A^{B|\Lambda} &= -2\sum_{i=1}^{k} g_i\,\left(e^{-\mathrm{i}\theta_i}\varphi_1 - e^{\mathrm{i}\theta_i}\varphi_2\right)\,\Big(\,f_{\Lambda_i\Sigma_i}{}^{\Lambda}\,\Phi^{\Lambda_i}_{AC}\,\Phi^{CB|\Sigma_i} + \\
&\qquad\qquad \eta_{\Delta_i\Omega_i}\,f_{\Lambda_i\Sigma_i}{}^{\Delta_i}\,\Phi^{\Lambda}_{CD}\,\Phi^{CD|\Omega_i}\,\Phi^{\Lambda_i}_{AE}\,\Phi^{EB|\Sigma_i}\,\Big) \ ,
\end{aligned} \tag{4.23}$$

where g_i are the coupling constants associated to each simple factor in the full gauge group and the constant phases θ_i are additional parameters, also in one–to–one correspondence with such simple factors.

4.2.1 Solvable Lie algebra parametrization of the scalar manifold

The choice of a convenient parametrization of the scalar manifold might simplify computations and help to characterize the geometrical meaning of the various fields. The so-called solvable Lie algebra parametrization has proven useful in many questions concerning extended supergravities [53, 54, 55], and we shall apply it also to the scalar manifold of standard $\mathcal{N}=4$ supergravity (4.16), as this will prove useful in the following.

In the solvable Lie algebra approach, the non-compact scalar manifold G/H is obtained exponentiating an appropriate solvable subalgebra $\mathrm{Solv}(G/H)$ of the isometry algebra $\mathbb{G}$ (for a review see [56]):

$$\frac{G}{H} \simeq \exp\left(\mathrm{Solv}\left(\frac{G}{H}\right)\right) \ . \tag{4.24}$$

The solvable Lie algebra $\mathrm{Solv}(G/H)$ is spanned by the non compact Cartan generators $\mathcal{H}_i$ of $\mathbb{G}$ plus all the step operators E^α associated with roots α that are not orthogonal to all the non compact Cartan generators. This approach has the distinctive advantage of giving a unique group theoretical characterization to all the scalar fields ϕ of supergravity and provides simple polynomial parametrizations of the coset representatives $L(\phi) \in G$.

Let us now discuss the solvable parametrization of the manifold (4.16).

The solvable Lie algebra of the coset $\mathbf{SU(1,1)/U(1)}$ We write the three generators of the $\mathrm{sl}(2,\mathbb{R})$ Lie algebra in the form:

$$l_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \ , \ l_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ l_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \tag{4.25}$$

leading to the standard form of the commutation relations:

$$[l_0, l_\pm] = \pm\, l_\pm \quad , \quad [l_+\, l_-] = 2\, l_0 \ . \tag{4.26}$$

The solvable Lie algebra generating the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ can be taken to be spanned by the two non compact generators l_0, l_- and we can write the coset element in the form:

$$\Lambda(\alpha,\beta) = \exp(\alpha\, l_0)\, \exp(\beta\, l_-) \ = \begin{pmatrix} \mathrm{e}^{\frac{\alpha}{2}} & 0 \\ \beta\, \mathrm{e}^{-\frac{\alpha}{2}} & \mathrm{e}^{\frac{-\alpha}{2}} \end{pmatrix} \ . \tag{4.27}$$

We can convert an $\mathrm{SL}(2,\mathbb{R})$ matrix Λ into an $\mathrm{SU}(1,1)$ matrix $\mathcal{S} = \mathcal{C}\Lambda\mathcal{C}^{-1}$ by means of a Cayley transformation:

$$\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \ , \tag{4.28}$$

and we obtain the solvable parametrization of the coset representative (4.18)

$$\mathcal{S} \equiv \begin{pmatrix} \varphi_1 & \varphi_2^* \\ \varphi_2 & \varphi_1^* \end{pmatrix} = \begin{pmatrix} \cosh\frac{\alpha}{2} - \frac{\mathrm{i}}{2}\,\beta\,\mathrm{e}^{-\alpha/2} & \sinh\frac{\alpha}{2} - \frac{\mathrm{i}}{2}\,\beta\,\mathrm{e}^{-\alpha/2} \\ \sinh\frac{\alpha}{2} + \frac{\mathrm{i}}{2}\,\beta\,\mathrm{e}^{-\alpha/2} & \cosh\frac{\alpha}{2} + \frac{\mathrm{i}}{2}\,\beta\,\mathrm{e}^{-\alpha/2} \end{pmatrix} \tag{4.29}$$

that enters the potential and fermion shift formulae of standard $\mathcal{N}=4$ supergravity (4.23).

The solvable Lie algebra of the coset $\mathbf{SO(6,3)/SO(6)\times SO(3)}$ We should now consider the second factor in the scalar manifold (4.16), namely $\mathrm{SO}(6,n)$ modded by the action of the subgroup $\mathrm{SO}(6)\times\mathrm{SO}(n)$. The rank (i.e., the number of non–compact generators) of this non–compact coset is 3 and its dimension is $6n$. This means that the corresponding solvable Lie algebra is spanned by 3 Cartan generators and $6n-3$ step operators E^α. Anticipating a result that will be discussed in the next section, since our main interest is in the partial breaking mechanism from $\mathcal{N}=4$ to $\mathcal{N}=3$ we focus on the case $n=3$ which suited to describe the minimal gauging that triggers such a breaking. Most of the following formulae can in any case immediately be generalized to any n.

A generic element $\mathcal{A}$ of the Lie algebra $\mathrm{so}(6,3)$, using the Killing metric (4.20), has the following 3×3 block structure:

$$\mathcal{A} = \left(\begin{array}{c|c||c} A_3 & -B_3^T & B_2^T \\ \hline B_3 & A_2 & B_1^T \\ \hline\hline B_2 & -B_1 & A_1 \end{array}\right) \tag{4.30}$$

where $A_i = -A_i^T$ $(i=1,2,3)$ span an $\mathrm{so}(3)_1\times\mathrm{so}(3)_2\times\mathrm{so}(3)_3$ subalgebra of $\mathrm{so}(6,3)$, while the B_i are generic. The orthogonal splitting $\mathbb{G}=\mathbb{H}\oplus\mathbb{K}$ of this algebra with respect to the subalgebra $\mathbb{H}\equiv\mathrm{SO}(3)\times\mathrm{SO}(6)$ is as follows:

$$\mathbb{H}\ni\left(\begin{array}{c|c||c} A_3 & -B_3^T & 0 \\ \hline B_3 & A_2 & 0 \\ \hline\hline 0 & 0 & A_1 \end{array}\right)\ , \quad \mathbb{K}\ni\left(\begin{array}{c|c||c} 0 & 0 & B_2^T \\ \hline 0 & 0 & B_1^T \\ \hline\hline B_2 & -B_1 & 0 \end{array}\right)\ . \tag{4.31}$$

The solvable presentation of the coset is obtained by choosing coset representatives $L=\exp(\mathrm{Solv}(G/H))$, instead of the choice $L_{\mathrm{orth}}=\exp(\mathbb{K})$ leading to the standard off-diagonal parametrization. The structure of the solvable Lie algebra is the following.

The three-dimensional non-compact Cartan subalgebra contains the matrices of the form

$$\left(\begin{array}{c|c||c} 0 & 0 & 0 \\ \hline 0 & 0 & H \\ \hline\hline 0 & H & 0 \end{array}\right)\ , \tag{4.32}$$

where $H=\mathrm{diag}(h_1,h_2,h_3)$.

The remaining 15 dimensional space, spanned by the step operators associated with positive roots that are not orthogonal to the non–compact Cartan subalgebra, is given by the matrices either of the form

$$\left(\begin{array}{c|c||c} 0 & -E^T & E^T \\ \hline E & 0 & 0 \\ \hline\hline E & 0 & 0 \end{array}\right)\ , \tag{4.33}$$

with E a generic 3×3 matrix (this gives 9 parameters), or of the form

$$\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & U - U^T & U^T + D^T \\ \hline 0 & U + D & D^T - D \end{array}\right) , \tag{4.34}$$

where U is upper triangular and D lower-triangular, accounting for 6 parameters.

The entries of the matrices H, E, U, D can be used as the coordinates of the 18-dimensional space containing the scalar fields of the 3 vector multiplets that, as we shall see, participate to the $\mathcal{N} = 4 \to \mathcal{N} = 3$ supersymmetry breaking.

4.3 Partial breaking to $\mathcal{N} = 3$

If we consider the $\mathcal{N} = 3$ massive spin–3/2 multiplet of eq. (3), it is easy to be convinced that the minimal number of vector multiplets one has to couple to $\mathcal{N} = 4$ supergravity in order to produce a spontaneous breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 3$ is three.

Indeed for each massive gravitino we have 6 massive vectors and 14 massive scalars. In the super-Higgs mechanism, the 6 vectors become massive by eating up as many scalars: this means that, including the 3 graviphotons that remain massless in $\mathcal{N} = 3$ supergravity, we need 20 scalars and 9 vectors to begin with. This counting is in agreement with the coupling of 3 vector multiplets of $\mathcal{N} = 4$ supersymmetry: the 9 vectors are the 6 graviphotons of $\mathcal{N} = 4$ supergravity plus the 3 matter photons. The 20 scalars are the 6×3 matter scalars plus the 2 scalars contained in the $\mathcal{N} = 4$ graviton multiplet. The counting agrees also at the fermion level. Before breaking we have 4 dilatinos plus $4 \times 3 = 12$ gauginos, a total of 16 spin–1/2 fermions. After breaking we remain with 1 dilatino sitting in the $\mathcal{N} = 3$ graviton multiplet and 14 spinors sitting in the massive gravitino multiplet, a total of 15. The missing fermion is the *goldstino* which has been eaten up by the gravitino to become massive.

A more detailed analysis that includes also the isospin representation assignments displayed in table 3 gives additional information. In order to realize the partial breaking $\mathcal{N} = 4 \to \mathcal{N} = 3$ in AdS$_4$, the three $\mathcal{N} = 4$ vector multiplets plus the 6 graviphotons must be used to gauge a minimal group:

$$\mathcal{G}_{\min} = \mathrm{SO}(4) \times \mathrm{SO}(3) = \mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2 \times \mathrm{SO}(3)_3 \ . \tag{4.35}$$

In AdS$_4$ the relevant supersymmetry algebra is $\mathrm{Osp}(\mathcal{N}|4)$ and the gravitinos are assigned to the vector $\mathcal{N}$–dimensional representation of the R–symmetry subalgebra $\mathrm{SO}(\mathcal{N}) \subset \mathrm{Osp}(\mathcal{N}|4)$. With respect to the $\mathrm{SU}(\mathcal{N})$ local symmetry group of the supergravity Lagrangian, the embedding of $\mathrm{SO}(\mathcal{N})$ is the maximal one described by selecting those unitary, uni-modular $\mathcal{N} \times \mathcal{N}$ matrices that are also real and hence orthogonal. The partial breaking $\mathcal{N} = 4 \to \mathcal{N} = 3$ is algebraically described by

the embedding of Osp(3|4) into Osp(4|4): in particular the R–symmetry subgroup of $\mathrm{SO}(3) \subset \mathrm{Osp}(3|4)$ is the subgroup of $\mathrm{SO}(4) \subset \mathrm{Osp}(4|4)$ such that the vector representation decomposes as $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$, the singlet being the massive gravitino, the triplet containing the massless ones. When we look at SO(4) as $\mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2$, this embedding corresponds to choosing $\mathrm{SO}(3)_R = \mathrm{diag}(\mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2)$.

On the other hand, a look at table 3 shows that also the massive vectors sitting in the massive gravitino multiplet are in the triplet of the R–symmetry group $\mathrm{SO}(3)_R$. This implies that the three matter multiplets coupled to $\mathcal{N} = 4$ supergravity must gauge a third SO(3) group and that the residual unbroken R–symmetry group must be the diagonal subgroup of the three:

$$\mathrm{SO}(3)_R = \mathrm{diag}\ (\mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2 \times \mathrm{SO}(3)_3)\ . \tag{4.36}$$

So this argument uniquely selects the model that in standard $\mathcal{N} = 4$ supergravity can realize the spontaneous breaking in AdS_4 to an $\mathcal{N} = 3$ theory with a residual $\mathcal{G}'$ unbroken gauge symmetry and one massive gravitino multiplet. It is the gauging of the group

$$\mathcal{G} = \mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2 \times \mathrm{SO}(3)_3 \times \mathcal{G}'\ . \tag{4.37}$$

4.3.1 Moduli space of $\mathcal{N} = 3$ vacua

Among the vector multiplet scalars, only those of the 3 multiplets gauging the $\mathrm{SO}(3)_3$ group in (4.37) can actually be relevant for the partial breaking mechanism. These 18 fields span the coset $\mathrm{SO}(6,3)/(\mathrm{SO}(6) \times \mathrm{SO}(3))$, whose parametrization we already discussed in sec. 4.2.1. The residual unbroken gauge group $\mathcal{G}'$ is a mere spectator in this game, and we can forget about the associated scalar fields.

On the other hand, recalling table 3, we know that, after partial breaking, the massive gravitino multiplet contains a total of 4 scalar (or pseudo-scalar) fields that are singlet under the $\mathrm{SO}(3)_R$ R–symmetry group of $\mathcal{N} = 3$ supergravity. These are the only fields that can develop non vanishing v.e.vs in an $\mathcal{N} = 3$ vacuum and constitute the moduli space $\mathcal{M}^{\mathrm{p.b.}}$ we are looking for, and that was anticipated in eq. (4.2).

Two of these fields are, of course, the scalars of the $\mathcal{N} = 4$ graviton multiplet, spanning the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space that was already discussed. The other two moduli are the $\mathrm{so}(3)_R$-singlets among the 18 vector-multiplet scalars. They belong to the sub-manifold of $\mathrm{SO}(6,3)/(\mathrm{SO}(6) \times \mathrm{SO}(3))$ spanned by the solvable Lie algebra generators that commute with the diagonal subgroup (4.36). Referring to eq. (4.30), we see that the matrices of the $\mathrm{so}(3)_R$ subalgebra are obtained setting $A_1 = A_2 = A_3 = A$ and $B_1 = B_2 = B_3 = 0$. The normalizer of this subalgebra in $\mathrm{so}(3,6)$ is an $\mathrm{sl}(2,\mathbb{R})$ Lie algebra spanned by the following three matrices:

$$l_0 = \left(\begin{array}{c|c||c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline\hline \mathbf{0} & \mathbf{1} & \mathbf{0} \end{array}\right)\ , \quad l_- = \left(\begin{array}{c|c||c} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \hline -\mathbf{1} & \mathbf{0} & -\mathbf{0} \\ \hline\hline \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}\right)\ , \quad l_+ = \left(\begin{array}{c|c||c} \mathbf{0} & -\mathbf{1} & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline\hline \mathbf{1} & -\mathbf{0} & \mathbf{0} \end{array}\right)\ , \tag{4.38}$$

that satisfy the $\mathrm{sl}(2,\mathbb{R})$ commutation relations in the standard form (4.26). If we exponentiate the solvable Lie algebra spanned by l_0 and l_- we generate a coset, locally isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$, embedded into $\mathrm{SO}(6,3)/(\mathrm{SO}(6)\times\mathrm{SO}(3))$, whose coordinates are R–symmetry singlets and can develop non vanishing v.e.vs in the $\mathcal{N}=3$ vacua. This coset is the second factor, $\mathrm{SO}(2,1)/\mathrm{SO}(2)$, in the moduli space $\mathcal{M}^{\mathrm{p.b.}}$ of $\mathcal{N}=3$ vacua, eq. (4.2).

According to this analysis, we can introduce an $\mathrm{SO}(6,3)$ coset representative $\widehat{L}$ defined, in a similar way to eq.(4.27), by

$$L = \exp(a\, l_0)\, \exp(b\, l_-) \ . \tag{4.39}$$

To better characterize the coset, it is however convenient to perform a change of basis $\mathcal{B}$ such as to bring the Killing form η of eq. (4.20) into

$$\eta' = \mathcal{B}\eta\mathcal{B}^{-1} = \mathrm{diag}(+,-,-,+,-,-,+,-,-) \ . \tag{4.40}$$

The group $\mathrm{SO}(3,6)$ admits a $\mathrm{SO}(1,2)^3$ subgroup whose matrices are block-diagonal in the rotated basis. The $\mathrm{sl}(2,\mathbb{R}) \sim \mathrm{so}(2,1)$ algebra of eq. (4.38) is the Lie algebra of the diagonal subgroup

$$\mathrm{diag}\ (\,\mathrm{SO}(1,2)_1 \times \mathrm{SO}(1,2)_2 \times \mathrm{SO}(1,2)_3) \ . \tag{4.41}$$

The coset representative (4.39) in the new basis becomes thus block-diagonal:

$$L' = \mathcal{B}\, L\, \mathcal{B}^{-1} = \mathrm{diag}(\mathcal{L},\mathcal{L},\mathcal{L}) \ . \tag{4.42}$$

It is made of three copies of the same $\mathrm{SO}(1,2)$ matrix

$$\mathcal{L} = \begin{pmatrix} \frac{1+b^2+\mathrm{e}^{2a}}{2\,\mathrm{e}^a} & \frac{-1+b^2+\mathrm{e}^{2a}}{2\,\mathrm{e}^a} & \frac{b}{\mathrm{e}^a} \\ \frac{-1-b^2+\mathrm{e}^{2a}}{2\,\mathrm{e}^a} & \frac{1-b^2+\mathrm{e}^{2a}}{2\,\mathrm{e}^a} & -\frac{b}{\mathrm{e}^a} \\ b & b & 1 \end{pmatrix} \ . \tag{4.43}$$

The above $\mathrm{SO}(1,2)$ matrix is quadratic in the entries $\mathrm{e}^{\pm a/2}$ and $b\mathrm{e}^{-a/2}$ that would appear in a $\mathrm{SL}(2,\mathbb{R})$ matrix constructed analogously to (4.27). This is quite natural since the 2×2 $\mathrm{SL}(2,\mathbb{R})$ matrices provide the spinor representation of $\mathrm{SO}(1,2)$, while the 3×3 matrix (4.43) is in the vector representation. This shows that in the matter sector the coset generated by eq. (4.39) is better interpreted as the coset $\mathrm{SO}(1,2)/\mathrm{SO}(2)$.

This also suggests the use of a different parametrization that somehow simplifies the expression of the coset representative. Indeed the $\mathrm{SO}(1,2)/\mathrm{SO}(2)$ coset is an instance of an off–diagonal non–compact coset for which we can use a standard projective parametrization. A coset representative in this parametrization reads

$$\mathcal{L}^{(\mathrm{proj})} = \frac{1}{\sqrt{1-v^2}} \left(\begin{array}{c|c} 1 & \mathbf{v}^T \\ \hline \mathbf{v} & \sqrt{1-v^2}\,\mathbf{1} + \frac{\mathbf{v}\mathbf{v}^T}{v^2}\left(1-\sqrt{1-v^2}\,\right) \end{array} \right) \ , \tag{4.44}$$

where the squared norm $v^2 \equiv \mathbf{v}^T\mathbf{v}$ of the vector $\mathbf{v} = (v_1, v_2)$ is bounded: $v^2 < 1$. The relation between the projective coordinates v_1, v_2 and the coordinates a, b of the solvable parametrization is found to be the following:

$$\mathrm{e}^a = \pm\frac{1+v_1}{\sqrt{1-v_1^2-v_2^2}} \ , \quad b = \pm\frac{v_2}{\sqrt{1-v_1^2-v_2^2}} \ , \tag{4.45}$$

where either both plus or both minus signs are taken. The sign ambiguity corresponds just to the usual double covering of the spinor representation.

4.3.2 Discussion of $\mathcal{N} = 3$ vacua and moduli dependence of E_0

Our final goal is to discuss the manifold of $\mathcal{N} = 3$ vacua of standard $\mathcal{N} = 4$ supergravity and to derive the moduli dependence of the scale dimension E_0, see eq. (4.3). What we have to do is just to derive the dependence of the fermion shifts and mass matrices on the 4 scalar fields that are R–symmetry singlets and impose the conditions for $\mathcal{N} = 3$ unbroken supersymmetry. This can in principle single out a sub-manifold of the moduli space $\mathcal{M}^{\mathrm{p.b.}}$ or impose restrictions on the additional parameters g_i and θ_i appearing in eq. (4.23). We will then investigate the dependence of E_0 on the fields and parameters that remain free.

Hence, according to the discussion above, we introduce a new coset representative for $\mathrm{SO}(3,6)$ which is generated as follows:

$$L^{(\mathrm{proj})} = \mathcal{B}^{-1}\,\mathrm{diag}\left(\mathcal{L}^{(\mathrm{proj})}, \mathcal{L}^{(\mathrm{proj})}, \mathcal{L}^{(\mathrm{proj})}\right)\,\mathcal{B} \ . \tag{4.46}$$

The next point in the explicit search of $\mathcal{N} = 3$ vacua involves the calculation of three complex quantities, introduced by Wagemans and de Roo [14, 46] and named Z_1, Z_2, Z_3. They have the following definition in terms of the matrix elements of the coset representative:

$$\begin{aligned} Z_1 &= \mathcal{L}^{(\mathrm{proj})}_{4,1} + \mathrm{i}\,\mathcal{L}^{(\mathrm{proj})}_{7,1} \ , \\ Z_2 &= \mathcal{L}^{(\mathrm{proj})}_{4,4} + \mathrm{i}\,\mathcal{L}^{(\mathrm{proj})}_{7,4} \\ Z_3 &= \mathcal{L}^{(\mathrm{proj})}_{4,7} + \mathrm{i}\,\mathcal{L}^{(\mathrm{proj})}_{7,7} \ . \end{aligned} \tag{4.47}$$

A simple explicit expression of these quantities is obtained introducing a polar parametrization of the vector $\mathbf{v}$ of the projective presentation (4.44). As noticed after eq. (4.44), the vector $\mathbf{v}$ ranges in the Poincarè disk $\Delta \approx \mathrm{SO}(1,2)/\mathrm{SO}(2)$. Hence we set

$$v_1 + \mathrm{i}v_2 = \rho\,\mathrm{e}^{\mathrm{i}\theta} \tag{4.48}$$

and we obtain:

$$\begin{aligned} Z_1 &= \frac{\rho}{\sqrt{1-\rho^2}}\,\mathrm{e}^{\mathrm{i}\theta} \ , \\ Z_2 &= \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{1-\rho^2}}\right) - \left(1-\frac{1}{\sqrt{1-\rho^2}}\right)\mathrm{e}^{2\mathrm{i}\theta}\right] \ , \end{aligned}$$

$$Z_3 = \frac{\mathrm{i}}{2}\left[\left(1+\frac{1}{\sqrt{1-\rho^2}}\right)+\left(1-\frac{1}{\sqrt{1-\rho^2}}\right)\mathrm{e}^{\mathrm{i}\theta}\right] . \tag{4.49}$$

As we see, in the origin of the coset manifold, at $w = 0$ we have $Z_1 = 0, Z_2 = 1, Z_3 = \mathrm{i}$.

As previously discussed, see eq. (4.12), to preserve $\mathcal{N} = 3$ unbroken supersymmetries it is necessary to fix three out of the four eigenvalues of the fermion shift matrices. As shown by de Roo and Wagemans [14], these conditions are satisfied when the three following quantities are equal:

$$g_1 Z_1\left(\mathrm{e}^{\mathrm{i}\theta_1}\varphi_1^\star+\mathrm{e}^{-\mathrm{i}\theta_1}\varphi_2^\star\right) = g_2 Z_2\left(\mathrm{e}^{\mathrm{i}\theta_2}\varphi_1^\star+\mathrm{e}^{-\mathrm{i}\theta_2}\varphi_2^\star\right) = g_3 Z_3\left(\mathrm{e}^{\mathrm{i}\theta_3}\varphi_1^\star+\mathrm{e}^{-\mathrm{i}\theta_3}\varphi_2^\star\right) \equiv \Theta . \tag{4.50}$$

In this case, the mass of the three gravitinos gauging the unbroken supersymmetries is

$$M_{(3)} = |\Theta| , \tag{4.51}$$

while the mass of the "broken gravitino" is:

$$M_{(1)} = |\Theta|\sqrt{\frac{8+\rho^4-\rho^2(8+\rho^2)\cos 4\theta}{8+\rho^4-8\rho^2-\rho^4\cos 4\theta}} . \tag{4.52}$$

The ratio of the two masses does not depend on the vev.s of the scalars in the graviton multiplet nor on the three free parameters θ_i, but only on the variables ρ and θ that parametrize the matter coset. It is straightforward to show that the ratio is bounded by three and that this particular value is never reached in the bulk of the moduli space. On the contrary, three is the limit of the mass ratio at the boundary of the disk. Figure 2 represents the projection of the moduli space on the sub-factor Δ. Each point of the disk, namely each possible couple of vev.s for the matter scalars, determines through eq. (4.50) a specific set of values for the remaining vev.s and parameters, that preserve $\mathcal{N} = 3$ unbroken supersymmetries. The lines of fixed ratio $M_{(1)}/M_{(3)}$ are shown. Figure 3 gives a three dimensional view of the same plot. It is now interesting to remark the meaning of the ratio $M_{(1)}/M_{(3)}$. Indeed for the normalization conventions of de Roo and Wagemans adopted so far, the mass matrix eigenvalues M of the gravitinos linearly depend on their Casimir energy $E_{(3/2)}$ (see eq. 3.28):

$$M = \left|m_{(3/2)}+4\right| e = (4E_{(3/2)}-6)\, e . \tag{4.53}$$

Now, the energy of the "massless" (i.e. $m_{(3/2)} = 0$) gravitinos is fixed (in units of the Freund Rubin scale, e): $E^{\mathrm{massless}}_{(3/2)} = 5/2 \;\Rightarrow\; M_{(3)} = 4$. Hence the ratio

$$\frac{M_{(1)}}{M_{(3)}} = \frac{4E_{(3/2)}-6}{4} = E_{(3/2)}-\frac{3}{2} = E_0 \tag{4.54}$$

exactly gives the Clifford vacuum energy of the massive "unbroken gravitino" multiplet. The most interesting thing to note about this point, is that the boundary

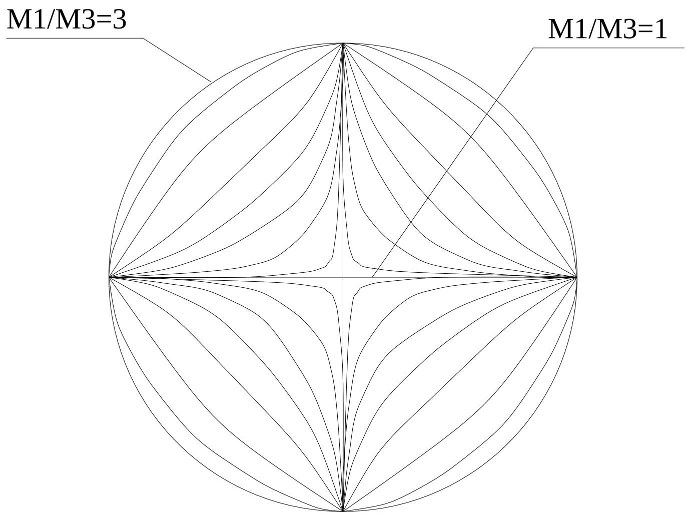

Figure 2: Projection of the moduli space $\mathcal{M}^{\text{p.b.}}$ onto the disk $\Delta \approx \mathrm{SO}(1,2)/\mathrm{SO}(2)$. The lines of constant ratio $M_{(1)}/M_{(3)}$ are shown.

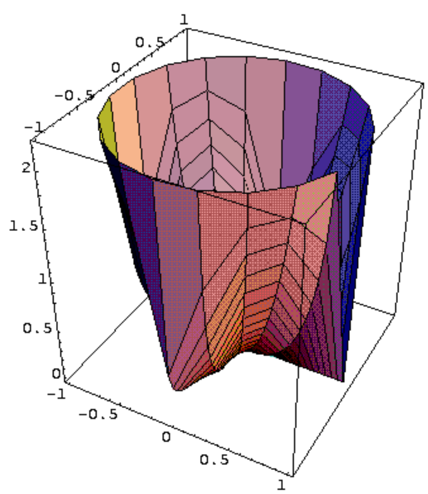

Figure 3: A three dimensional plot of the ratio $M_{(1)}/M_{(3)}$ over the disk $\Delta \approx \mathrm{SO}(1,2)/\mathrm{SO}(2)$

value $E_0 = 3$ is exactly the energy of the shadow multiplet of the $N^{0,1,0}$ compactification. This means that the dimensional reduction of $D = 11$ supergravity on this background does not belong to the family of $\mathcal{N} = 4$ four-dimensional supergravities known in the literature.

4.4 Comparison with Kaluza Klein theory

To be more precise about this last statement, we want now to derive the effective four dimensional Lagrangian from the eleven dimensional supergravity theory

compactified on the $N^{0,1,0}$ coset space, taking into account (some of) the modes belonging to the "shadow" massive gravitino multiplet. As already said, we expect such a truncation to be a consistent one. Moreover, since the only consistent way to give mass to a gravitino is via a super-Higgs mechanism, we expect that the effective 4D theory actually be a $\mathcal{N}=4$ supergravity, of which the $N^{0,1,0}$ solution represent a $\mathcal{N}=3$ extremum.

In order to make a complete comparison between Kaluza Klein theory and the standard broken $\mathcal{N}=4$ supergravity, we also need the form in this latter of the kinetic terms of the scalar fields parametrizing the moduli space $\mathcal{M}^{\mathrm{p.b.}}$ of eq. (4.2). The two factors in eq. (4.2) both correspond to a copy of the upper half plane. The σ-model metric for the scalars in each factor will therefore be the Poincaré metric.

In the solvable parametrization we adopt, this is seen as follows. Let us focus on one of the factors, e.g. $\mathrm{SU}(1,1)/\mathrm{U}(1)$, containing the scalars of the graviton multiplet; the formulae for the other factor will be analogue. In terms of the coset representative (4.27) we calculate the left-invariant 1–form

$$\Omega = \Lambda^{-1} d\Lambda = \begin{pmatrix} \frac{1}{2} d\alpha & 0 \\ -\beta\, d\alpha + d\beta & -\frac{1}{2} d\alpha \end{pmatrix} \tag{4.55}$$

and then we calculate the vielbein projecting onto the normalized non–compact generators that span the subspace $\mathbf{k}$ orthogonal to the compact subalgebra $\mathbf{h}$. We obtain the zweibein 1–form corresponding to the metric

$$ds^2 = \frac{d\alpha^2 + \left(-\left(\beta\, d\alpha\right) + d\beta\right)^2}{2} \ . \tag{4.56}$$

The metric (4.56) is exactly in the form of the standard Poincaré metric

$$ds^2 = \frac{dS\, d\bar{S}}{(\mathrm{Im} S)^2} \tag{4.57}$$

on the upper half plane, upon setting

$$S = \left(-\mathrm{i} e^{-\alpha} + \beta\, e^{-\alpha}\right)^{-1} \ . \tag{4.58}$$

Looking also at the kinetic terms of the vector fields in Wagemans and de Roo action [13, 52], one can see that S is exactly the *dilaton* field, namely it appears as a generalized field dependent coupling constant in the standard way. By this we mean that with the identification (4.58) the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ of the vector fields has the standard form

$$\mathcal{N} = \mathrm{Re} S\, \eta + \mathrm{i}\, \mathrm{Im} S\, \eta\, L\, L^T\, \eta \tag{4.59}$$

obtained through the Gaillard Zumino formula and through the standard embedding of $SL(2,R) \times SO(6,n)$ into $Sp(2(n+6),R)$ as described in lectures [57] (see eq. (105) of that paper).

The metric for the R–symmetry neutral vector multiplet scalars that fill the $\mathrm{SO}(2,1)/\mathrm{O}(2)$ factor in the moduli space $\mathcal{M}^{\mathrm{p.b.}}$ is another copy of the Poincaré metric (4.56) with (a,b) replacing (α,β).

4.4.1 Effective 4-dimensional theory for $N^{0,1,0}$ compactification

To derive the effective theory for the $N^{0,1,0}$ compactification, we consider a minimal set-up consisting of the inclusion only of the shadow massive scalars which are singlets of the $\mathcal{N}=3$ R-symmetry group $\mathrm{SU}(2)_R$; let us denote them collectively by μ. We shall in particular derive the σ-model kinetic terms of the scalars as well as their potential $V(\mu)$.

We discussed in section 3.3.1 the singlet shadow scalars and their geometric interpretation as moduli of the $N^{0,1,0}$ compactification. They are the "breathing" and "squashing" modes $\Sigma(x)$ and $\phi(x)$, of energies 6 and 4, and the modes $\pi_1(x)$ and $\pi_2(x)$ related to an internal 3-form, of energies 5 and 3. These fields should correspond to mass eigenstates of the mass matrix in the effective theory expanded around the $N^{0,1,0}$ vacuum.

Including the fields Σ and ϕ corresponds to rescaling as in eq. (3.68) the 7-dimensional vielbein. The effective theory that includes rescalings of the internal vielbein has been already considered in [11], with the aim of checking the stability of the compactification. The contributions from the internal 3-form modes, eq. (3.79), were not considered in that paper.

The basic techniques for the "warped" dimensional reduction, corresponding to eq.s (3.68) and (3.79), of the 11D supergravity Lagrangian (2.1) go back to [58]. To obtain a 4D supergravity in the Einstein frame, it is necessary to perform a Weyl rescaling of the 4D metric. Thus we set

$$V^a(x) = \mathrm{e}^{21\,\Sigma(x)}\,E^a(x)\ , \tag{4.60}$$

the vierbein E^a corresponding to the Einstein frame metric.

The kinetic part of the 4-dimensional action arise both from the Einstein-Hilbert term in the 11D action (2.1) under the rescalings (3.68) and from the "Yang-Mills" part F^2_{MNPQ} because of the mixed terms $F_{a\beta\gamma\delta}$ due to the internal 3-form (3.79). To exhibit the σ-model kinetic terms corresponding to the expected structure, eq. (4.2), of the moduli space we leave the energy eigenstates $\Sigma(x), \phi(x), \pi_1(x), \pi_2(x)$ in favor of the combinations $w(x)$ and $z(x)$, given in eq. (3.70), of the rescalings, and of the fields $f(x)$ and $g(x)$ of eq. (3.79).

Then the kinetic action is

$$\begin{aligned} S_{\mathrm{kin}} &= \frac{1}{\kappa_4^2}\int\Big[\mathcal{R} + \frac{1}{4}(\partial_a w)^2 + \frac{9}{4}\mathrm{e}^{-2\,w}(\partial_a f)^2 \\ &+ \frac{3}{2}(\partial_a z)^2 + \frac{27}{2}\mathrm{e}^{-2\,z}(\partial_a g)^2\Big]\det E\ , \end{aligned} \tag{4.61}$$

where the 4-dimensional gravitational coupling constant is defined in terms of the 11-dimensional one and of the volume V_7 of the "standard" $N^{0,1,0}$ space by

$$\kappa_{11}^2 = \kappa_4^2\,V_7\ . \tag{4.62}$$

The 4D potential for the scalars gets contributions from all the terms in the action (2.1). The contributions from the Einstein-Hilbert term due to the "internal"

curvature $\mathcal{R}^{(7)}$, describing the x-dependence of the volume of the internal space, were already computed in [11]. The kinetic terms for the 3-form, $(F_{MNRP})^2$, contribute both because of the internal three-form (3.79), yielding a term $(F_{\alpha\beta\gamma\delta})^2$, and of the Freund-Rubin term $(F_{abcd})^2$. In fact we insist on a Freund-Rubin ansatz

$$F_{abcd}(x) = Q(x)\,\varepsilon_{abcd}\ , \tag{4.63}$$

where F_{abcd} are the components of F along the vielbeins E^a. The 11D "Maxwell" equation of motion, eq. (2.3), requires that Q is no longer a constant, but rather that

$$Q(x) = \mathrm{e}^{-2\,z(x)-\frac{1}{3}w(x)}\Big(1+\frac{9}{4}(f(x)\,g(x)+\frac{g^2(x)}{2})\Big)\ e\ . \tag{4.64}$$

The above reduces to the usual FR ansatz, eq. (2.6), when $w = z = f = g = 0$. The "Chern-Simons" term in the action (2.1) contributes to the 4D scalar potential because of those terms that can be written as being proportional to

$$\varepsilon^{abcd}F_{abcd}\ \varepsilon^{\alpha_1\ldots\alpha_3\beta_1\ldots\beta_4}A_{\alpha_1\ldots\alpha_3}F_{\beta_1\ldots\beta_4}\ . \tag{4.65}$$

Altogether, the potential appearing in the 4-dimensional action,

$$S_{\rm pot} = -\frac{1}{\kappa_4^2}\int e^2\,V(\mu)\,\det E\ , \tag{4.66}$$

is the following:

$$\begin{aligned} V(w,z,f,g) &= -12\,\mathrm{e}^{-w-2z}\left(1+8\mathrm{e}^{w-z}-2\mathrm{e}^{2(w-z)}\right)+36\,\mathrm{e}^{-w-6z}\Big(1+\frac{9}{4}(fg+\frac{g^2}{2})\Big)^2 \\ &+\ 216\,\mathrm{e}^{-w-4z}\left(f^2+2fg+\left(1+6\mathrm{e}^{2(w-z)}\right)g^2\right) \\ &+\ 729\,\mathrm{e}^{-w-6z}(g^2+2fg)\Big(1+\frac{9}{4}(fg+\frac{g^2}{2})\Big)\ . \end{aligned} \tag{4.67}$$

Extracting the quadratic part of the potential (4.67) around the $N^{0,1,0}$ critical point $w = z = f = g = 0$ we find the mass terms

$$24\,e^2\,(w^2+4\,wz+16\,z^2)+4\,e^2\,(27f^2+8\,fg+\frac{8}{27}g^2)\ . \tag{4.68}$$

The eigenvalues of the mass matrix for the fields $w/\sqrt{2}$ and $\sqrt{3}z$, those that have canonical kinetic terms in the $N^{0,1,0}$ vacuum, see eq. (4.61), correspond to AdS masses[12]

$$m_\Sigma^2 = 320\,e^2\ , \qquad m_\phi^2 = 96\,e^2\ , \tag{4.69}$$

[12]The definition of AdS$_4$ mass m that we use is such that the free eq. of motion of a scalar $\varphi(x)$ is $(\Box+\mathcal{R}^{(4)}/3+m^2)\varphi = 0$. In the $N^{0,1,0}$ background this leads to the AdS squared mass m^2 being shifted of $32\,e^2$ with respect to the "usual" squared mass $\bar{m}^2$, appearing as $\frac{1}{2}\bar{m}^2\varphi^2$ in the Lagrangian.

namely to energies $E = 6$ and $E = 4$, as we expected from table 3. The mass eigenstates are indeed proportional respectively to the fields $\Sigma(x)$ and $\phi(x)$.

The eigenvalues of the mass matrix for the canonically normalized fields $3f/\sqrt{2}$ and $g/(3\sqrt{3})$ correspond to AdS masses

$$m^2_{\pi_1} = 192\, e^2 \ , \qquad m^2_{\pi_2} = 32\, e^2 \ , \tag{4.70}$$

again in agreement with our expectations: the eigenstates are proportional to $\pi_1(x)$ and $\pi_2(x)$, and correspond to energies $E = 5$ and $E = 3$.

4.4.2 Comparison

From the eleven dimensional compactification geometric point of view, we have switched on 4D fields corresponding to rescaling in a particular way (see eq. (3.71)) the internal vielbein and to turn on a three form potential in the internal dimensions. This 11D point of view has also suggested the combinations of scalar fields to use. Indeed it is natural to switch on independently those components of the three form which scale in a different way under the rescalings (3.71); as already noticed, this is what gave rise to the definitions of the fields f and g. Moreover if we want to pair each of them with the field that scales the relative component of the three form (3.79), we see that the pairing of f with w and of g with z has a natural geometric interpretation.

From the form of the effective action (4.61) this pairing is also evident. In particular the kinetic terms suggest us that these two copies of fields parametrize two manifold that are both locally isometric to $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$. Indeed we can interpret the kinetic terms as the metric of this coset space in the solvable parametrization, see eq. (4.56). So it is now natural to ask if this 4D Lagrangian suggested by M theory in 11D can be interpreted as a “standard” Lagrangian describing the supersymmetry breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 3$.

Having the same form of the kinetic terms in both the theories, what we have to do is to look at the scalar potentials[13]. If we do this we can find certain analogies between the two, but there is no way to match one with the other. So we notice that the matching with the known supersymmetric $\mathcal{N} = 4$ theory spontaneously broken to $\mathcal{N} = 3$ is impossible, and these latter are the only known theories where a massive gravitino is coupled to $\mathcal{N} = 3$ massless gravitons.

Now there are two possibilities. One is that the KK truncation to the massive gravitino is an inconsistent one, but as previously discussed there are strong arguments that this is not the case. The other is that new “shadow” supergravities do exist.

Indeed, in the KK compactification of M theory on the coset manifold $N^{0,1,0}$ we cannot find the $\mathcal{N} = 4$ theory realized at the massless level, since this coset does not have the right isometries. Instead, we can have an enhancement of symmetry

[13]The explicit form of the scalar potential in the “standard” de Roo and Wagemans’ theory is rather cumbersome and we do not report it explicitly here, see [46, 13, 14, 52]

if we take into consideration also some massive mode. Then we can argue that in order to find $\mathcal{N}=4$ supersymmetry realized at the massless level in this context we have to look for some change of topology of the internal manifold.

From the point of view of the effective four dimensional theory, this involves something that happens at the boundary of the moduli space, where some rescaling vanishes or goes to infinity. This is specular to what happens if we try to match the standard $\mathcal{N}=4$ 4D Lagrangian with the result of KK compactification. Also in that case, we have shown that the requested value of the Casimir energy of the gravitino is never reached but it is the limiting value on the boundary of the moduli space (of the matter fields).

We conclude that a new shadow supergravity has to be built and that, if this new theory has something to do with the standard one, then it can only be at the boundary of the two respective moduli spaces. At the boundary one can guess the relation between the scalar fields. Indeed it is easy to see that if we perform the change of parametrization (4.45) on the fields z and g the effective potential (4.67) is the same for both the choice of the sign at the limiting value of $|v|=1$. This is not true for the fields w and f. So we can guess that in the $\mathcal{N}=4$ theory parent to the $N^{0,1,0}$ compactification the fields z and g will be those parametrizing the matter coset $\mathrm{SO}(1,2)/\mathrm{SO}(2)$, while w and f will belong to the new massless graviton multiplet.

Moreover, we can argue that the change of topology requested for the interpolation between the $N^{0,1,0}$ compactification and this theory involves a singularity of the vielbein $B^{\widetilde{\alpha}}$, ($\widetilde{\alpha}=4,5,6,7$), as the match with standard $\mathcal{N}=4$ supergravity is possible only at the boundary of the moduli space of the matter fields (related to z and g) and the action of the rescaling on the vielbein is as in (3.71).

5 On the suggested shadow extensions of $\mathcal{N}\geq 4$ supergravities

To understand the true origin of the bound (4.1) on the conformal dimension of the broken gravitino field, we need to adopt a viewpoint we have already considered in our general discussion of partial supersymmetry breaking (see section 4.1). Namely we have to turn things around and consider the coupling of a massive $\mathcal{N}=3$ gravitino multiplet to $\mathcal{N}=3$ supergravity. Then we can uniquely determine the scalar manifold $\mathcal{M}_{3/2}^{\mathcal{N}=3}$ containing the 14 scalars of such a multiplet.

Indeed, from table 3 we know that, of these 14 scalars, 2 are SU(3)–singlets, while 12 sit in the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representations. The SU(3)–singlets parametrize the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$ that, within the $\mathcal{N}=4$ context, is absorbed by the graviton multiplet. The remaining 12 fields must parametrize a non–compact 12-dimensional coset G/H satisfying the following conditions:

1. The isotropy subgroup is $H=\mathrm{SU}(3)\times\mathrm{U}(1)$, since we look for a theory

with manifest $\mathcal{N} = 3$ supersymmetry and admitting $\mathcal{N} = 3$ vacua. Since $\dim H = 9$, this implies that $\dim G = 21$.

2. The coset generators $\mathbb{K}$ in the orthogonal splitting $\mathbb{G} = \mathbb{H} \oplus \mathbb{K}$ must transform in the $\mathbf{6}$ (or $\bar{\mathbf{6}}$) of SU(3), namely in the (conjugate) symmetric tensor representation.

The unique solution of these constraints is $G = \mathrm{Usp}(3,3)$. Thus altogether we have

$$\mathcal{M}_{3/2}^{\mathcal{N}=3} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{USp}(3,3)}{\mathrm{SU}(3) \times \mathrm{U}(1)} \ . \tag{5.1}$$

In sec. (4.3) we have discussed how the coupling of a massive $\mathcal{N} = 3$ gravitino to $\mathcal{N} = 3$ supergravity can be obtained from $\mathcal{N} = 4$ supergravity, gauging the minimal group $\mathcal{G}_{\min}$ of eq. (4.35), via a super-Higgs phenomenon. In this case, before breaking, the scalars fill the 20-dimensional[14] space $\mathcal{M}_{(3)}^{(\mathcal{N}=4)}$, see eq. (4.16).

If the super-Higgs mechanism of sec. (4.3) were really the most general way of coupling a massive gravitino multiplet to $\mathcal{N} = 3$ supergravity, we should be able to single out, within the scalar manifold of $\mathcal{N} = 4$ vector multiplets, namely the $\mathrm{SO}(3,6)/(\mathrm{SO}(3) \times \mathrm{SO}(6))$ factor in $\mathcal{M}_{(3)}^{(\mathcal{N}=4)}$, a consistent truncation to the scalar manifold of the gravitino multiplet (5.1). Indeed it should be possible to rewrite the theory in $\mathcal{N} = 3$ language with no additional restrictions and span the whole 14–dimensional moduli space of such a coupling.

But this is a crucial point: such a consistent truncation does not exist in standard $\mathcal{N} = 4$ supergravity, while it can be introduced if we take a different, $\mathcal{N} = 3$, starting point.

Solvable Lie algebra decompositions To discuss this point, it is convenient to use the solvable Lie algebras language. A non–compact coset manifold G_1/H_1 is consistently embedded as a sub-manifold of another non–compact coset G_2/H_2 iff the solvable Lie algebra $\mathrm{Solv}(G_1/H_1)$ generating the first coset is a subalgebra of the solvable Lie algebra $\mathrm{Solv}(G_2/H_2)$ generating the second coset.

It is then not difficult to see that the "matter" part $\mathrm{USp}(3,3)/(\mathrm{SU}(3) \times U(1))$ of the scalar manifold of the gravitino multiplet (5.1) can not be embedded into the scalar manifold of $\mathcal{N} = 4$ vector multiplets $\mathrm{SO}(3,6)/(\mathrm{SO}(3) \times \mathrm{SO}(6))$, since

$$\mathrm{Solv}\left(\frac{\mathrm{USp}(3,3)}{\mathrm{SU}(3) \times U(1)}\right) \not\subset \mathrm{Solv}\left(\frac{\mathrm{SO}(3,6)}{\mathrm{SO}(3) \times \mathrm{SO}(6)}\right) \ . \tag{5.2}$$

Indeed, the 12 generators that we should select among those of the solvable Lie algebra for $\mathrm{SO}(3,6)/(\mathrm{SO}(3) \times \mathrm{SO}(6))$, given in eq.s (4.32–4.34) are those of isospin

[14]Beside the two fields belonging to the $\mathcal{N} = 4$ graviton multiplet, 6 scalar are eaten up by as many vectors that become massive in the super-Higgs phenomenon.

$J = 2$ or isospin $J = 0$ with respect to the diagonal $\mathrm{SO}(3)_R$ R–symmetry group[15] (4.36). Such generators correspond to the matrices of eq. (4.32) (3 Cartan generators) or to the matrices of eq. (4.33) with the constraint $E = E^T$ (6 nilpotent generators) or, finally, to those of eq. (4.34) with $D = U^T$ (3 nilpotent generators). The problem is that these constraints are *not* preserved by the commutation relations, namely they do *not* define a proper subalgebra.

Let us instead suppose that the manifold of the scalars of the 3 vector multiplets be the one that arises in the coupling to $\mathcal{N} = 3$ supergravity, namely $\mathcal{M}_{(3)}^{(\mathcal{N}=3)}$ of eq. (4.17). This space *can* contain $\mathrm{USp}(3,3)/(\mathrm{SU}(3) \times U(1))$ as a sub-manifold, as

$$\mathrm{Solv}\left(\frac{\mathrm{USp}(3,3)}{\mathrm{SU}(3) \times U(1)}\right) \subset \mathrm{Solv}\left(\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)}\right) . \tag{5.3}$$

The solvable Lie algebra generating the manifold $\mathrm{SU}(3,3)/(\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1))$ was studied in detail in [55], and its structure is recalled in appendix C for the reader's convenience. It is immediate to select among the 18 generators listed in eq. (C.2) of that appendix those that belong to the subalgebra $\mathrm{USp}(3,3) \subset \mathrm{SU}(3,3)$. They are twelve:

$$g_i\,,\ h_i \quad (i = 1,2,3)\,,\ \mathbf{X}_1^+\,,\ \mathbf{Y}_1^+\,,\ \mathbf{Z}_1^+\,,\ \mathbf{X}_2^-\,,\ \mathbf{Y}_2^-\,,\ \mathbf{Z}_2^-\,, \tag{5.4}$$

and it is easy to check from eq. (C.1) that they close an algebra among themselves. It is the solvable Lie algebra generating $\mathrm{USp}(3,3)/\,(\mathrm{SU}(3) \times \mathrm{U}(1))$.

The consequence of the statements in eq.s (5.2) and (5.3) is that there should exist a *shadow extension* of $\mathcal{N} = 4$ supergravity constructed according to the recipe outlined below.

A conjecture about shadow $\mathcal{N} = 4$ supergravity We start from $\mathcal{N} = 3$ supergravity coupled to $3 + n$ vector multiplets. In this case the scalar manifold $\mathcal{M}_{(n)}^{(\mathcal{N}=3)}$ was given in eq. (4.17) and has an obvious $\mathrm{SU}(3,3)/(\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1))$ submanifold containing the essential degrees of freedom necessary for the super-Higgs phenomenon. Next we consider a *massless* $\mathcal{N} = 3$ gravitino multiplet, with the following field content:

Spin	Number of fields	SU(3) representation
$\frac{3}{2}$	1	$\mathbf{1}$
1	3	$\mathbf{3}$
$\frac{1}{2}$	3	$\overline{\mathbf{3}}$
0	2	$\mathbf{1}$

[15] This follows from the assignment to the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representations of SU(3) of the coset generators of $\mathrm{USp}(3,3)/(\mathrm{SU}(3) \times U(1))$ discussed above; indeed, $\mathrm{SO}(3)_R$ is maximally embedded into SU(3) and eq. (3.40) applies.

and we try to couple it to the theory we already have. The two scalar fields of the massless gravitino multiplet span the coset manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$ and in this way we have, altogether, the coset manifold:

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(3,3+n)}{\mathrm{SU}(3)\times\mathrm{SU}(n)\times\mathrm{U}(1)} \,. \tag{5.5}$$

Our conjecture is that, using the composite connections and the vielbein of the above coset (5.5), we should be able to construct a gauged supergravity action supersymmetric under $3+1$ local supersymmetries. This is the *shadow extension* of $\mathcal{N}=4$ supergravity we have many times invoked.

The catch of this construction, which we postpone to a future publication, is given by the $\mathrm{SU}(3)\times\mathrm{U}(1)$ charge assignments of the gravitinos: indeed these charges determine the coupling to the scalar sector, the gravitino mass–matrices and the fermion shifts.

In standard $\mathcal{N}=4$ supergravity, the three unbroken gravitinos and the fourth broken one sit together in the 4–dimensional fundamental representation of $\mathrm{SU}(4)$. When we split such a representation with respect to the $\mathrm{SU}(3)\times\mathrm{U}(1)$ subgroup, the $\mathrm{U}(1)$–charge of the singlet is 3 times the $\mathrm{U}(1)$ charge of the triplet. We think that this fact is at the origin of the bound (4.1) on the conformal dimension of the broken gravitino.

If we start instead from the coset manifold (5.5), there are two $\mathrm{U}(1)$ charges to be assigned, corresponding to the two $\mathrm{U}(1)$ factors appearing in the H–subgroup, and nothing a priori prevents us from independent assignments for the triplet and singlet gravitinos. Obviously, if our choices are incompatible with an $\mathrm{SU}(4)$ symmetry, we cannot expect any $\mathcal{N}=4$ supersymmetric vacuum from such a theory. But this is precisely what we are looking for: in all vacua the singlet gravitino will become massive by eating the goldstino and 6 vector fields will become its massive partners by eating up the 6 scalars of the 6-dimensional nilpotent subalgebra of $\mathrm{Solv}[\mathrm{SU}(3,3)/(\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1))]$. If our conjecture is true, it is quite obvious that the bound (4.1) on the conformal dimension of the broken gravitino will be replaced by a new one depending on the choice of the $U(1)$–charges, and that the critical value $E_0=3$ can now be reached for suitable choices.

Possible extensions to higher supergravities A comment that we should make in relation with our conjecture is that it might be applied also to the case of other extended supergravities, for instance $\mathcal{N}=8$.

Here the scalar manifold is $\mathcal{M}^{(\mathcal{N}=8)}=\mathrm{E}_{7(7)}/\mathrm{SU}(8)$, and the theory can be also regarded as the coupling of $\mathcal{N}=2$ supergravity to 15 $\mathcal{N}=2$ vector multiplets and 6 $\mathcal{N}=2$ massless gravitino multiplets. This gives an $\mathrm{SU}(2)\times\mathrm{SU}(6)\times\mathrm{U}(1)$ symmetry and the question is whether in the broken phase it is absolutely necessary to enforce the larger $\mathrm{SU}(8)$ local symmetry.

We are tempted to assume that the correct answer is no. In this case the 6 massive gravitinos can assume $\mathrm{U}(1)$–charges unrelated to those of the unbroken ones,

leading to less restrictive values for their conformal dimensions E_0. Eventually this destroys the $E_{7(7)}$ symmetry. Alternatively, we can break $\mathcal{N}=8$ into $\mathcal{N}=4$,$\mathcal{N}=5$ or $\mathcal{N}=6$ plus a complementary number of gravitino multiplets and formulate similar questions. This may be the solution of a puzzle recently discovered by Ferrara and Sokatchev [15]. It appears that ordinary $\mathcal{N}=8$ supergravity is able to produce only a subset of the possible BPS states allowed by Osp(8|4) representation theory, namely those obtained by tensoring only one kind of singleton representations while two are algebraically available. BPS states preserving a fraction of supersymmetry different from 1/2, 1/4 or 1/8 seem to be forbidden by the $E_{7(7)}$ duality symmetry of the theory. Their construction would involve the use of both kind of singletons. It is tempting to conjecture that these missing BPS states might be solutions of some *shadow extension* of $\mathcal{N}=8$ supergravity, constructed along the same lines of thought we have outlined above.

6 CFT interpretation of the spin $\frac{3}{2}$ shadow multiplet

In [3] we have already constructed the field theory realization, in the CFT dual of an $\mathcal{N}=3$ compactification, of the universal long gravitino multiplet which constitutes the main focus of the present paper. It is given by the following composite operator:

$$\mathcal{SH} = \mathrm{Tr}[\underbrace{\Theta_\Sigma \otimes \Theta_\Sigma \otimes \Theta_\Sigma}_{J=0}] = \mathrm{Tr}\left[\Theta^+_\Sigma \Theta^0_\Sigma \Theta^-_\Sigma\right] \,, \tag{6.1}$$

where Θ_Σ is the $J=1$ short superfield that, by definition, contains the field strength of the world–volume gauge theory:

$$\Theta_\Sigma = \begin{pmatrix} Y \\ \Sigma \\ -Y^\dagger \end{pmatrix} + \mathcal{O}(\theta^0)\,. \tag{6.2}$$

From the θ expansion of the superfield (6.1), we retrieve the field theory interpretation of the various Kaluza Klein modes appearing in the multiplet. As stressed in [3], the *breathing mode* of the internal manifold X^7, namely the scalar component of zero isospin and conformal dimension 6 corresponds to the following gauge theory operator:

$$\int d^2\theta^+ d^2\theta^- d^2\theta^0\, \mathcal{SH} = Tr\left[3\mathrm{i}H\overline{H}P + \tfrac{1}{4}\epsilon^{\lambda\mu\nu}\epsilon^{\rho\sigma\tau}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\tau}\right] + \text{derivative terms}\,, \tag{6.3}$$

where H and P are auxiliary fields of the world–volume gauge multiplet. In other words, the operator (6.3) is the $\mathcal{N}=3$ supersymmetrisation of the third power of

the gauge field strength [16]:

$$\epsilon^{\lambda\mu\nu}\epsilon^{\rho\sigma\tau}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\tau}\ ,$$

whose dimension, as a consequence of our analysis, is shown to be protected from quantum corrections. On the field theory side, this suggests the existence of some new no renormalization theorem yet to be discovered. A closely similar situation appears in type IIB AdS_5 compactifications, where the volume mode of the internal manifold corresponds to the CFT operator F^4, of dimension 8, which is known to satisfy some non-renormalization theorem. This consideration suggests that the operator (6.3) could originate by the low energy expansion of an analogue of the Dirac Born Infeld for the $M2$-brane, as well as the operator F^4 comes from the α' expansion of the DBI Lagrangian of the $D3$-brane (see[3]).

7 Conclusions and discussion

There are many directions where the analysis we started in this paper may lead. We already stressed that one basic suggestion of our work is the existence of shadow supergravities. Having exhaustively discussed this point in the previous sections, we can devote these conclusions to the quantum field theory aspects that originally motivated our work.

One immediate question is the fate of shadow multiplets in compactifications that are not of Freund-Rubin form. We known that for AdS_5, for example, interesting (both for supergravity and CFT) backgrounds [59] are not of Freund-Rubin form, having a non-trivial warp factor and internal antisymmetric tensor fields. If we could show that a pairing exists also for these compactifications, it would be interesting to study these cases as well and understand what changes in the form and the quantum numbers of the *universal* volume multiplet. Notice, for example, that the known $\mathcal{N}=4$ gauged supergravities admit a plethora of $\mathcal{N}=3$ vacua with $E_0<3$. It would be quite interesting to find the 11 dimensional solution corresponding to these critical points.

The $\mathcal{N}=4\to\mathcal{N}=3$ spontaneous symmetry breaking in four-dimensional gauged supergravity is also interesting because it provides examples of RG flows between 3d CFT's [10]. This is indeed the reason that originally motivated this work. The case $N^{0,1,0}$ is quite special. The critical point is at infinite distance from the origin in moduli space. Since the volume mode is singular (either vanishes or diverges), the $\mathcal{N}=4$ and $\mathcal{N}=3$ compactifications are related only through a change of topology. It is not completely clear what is the right interpretation from the quantum field theory point of view. We have made no attempt to identify the

[16] It should be noticed that, since vector multiplets are not conformal in three dimensions, the vector multiplet fields in the previous expression should be considered as re-expressed in terms of the fundamental degrees of freedom at the conformal point (via equations of motion or Hodge dualization, as requested by the specific example).

parent $\mathcal{N}=4$ theory in terms of the elementary degrees of freedom of the CFT associated with $N^{0,1,0}$, but this is certainly an aspect that deserves further interest.

We explicitly exhibited the form of the *universal* volume multiplet in terms of CFT operators in the case of $N^{0,1,0}$. It is obtained by tensoring three massless multiplets. It is the three-dimensional counterpart of the AdS_5 volume multiplet that contains an operator roughly of the form F^4, with dimension 8. The shadowing mechanism, which works also for AdS_5 compactifications, then guarantees that this multiplet has *canonical* dimension. This may be expected in view of non-renormalization theorems which are conjectured to hold for operators like F^4. We may speculate that something similar happens in three dimensions with the $E_0 = 6$ scalar operator. Since, in three dimensions, the CFT are not continuously connected to free theories but are just IR limit of some non-conformal gauge theory, it is difficult to give a general form for this *universal* operator. The explicit result reported in Section 6 might help.

When this paper was nearly finished we learnt of the very recent paper [60] where the question of consistent truncations of $AdS_4 \times X^7$ compactifications is addressed. It would certainly be interesting to inquire about the relation between consistency of the truncation and shadowing.

We leave all these interesting questions for future work.

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A Conventions and notations

We follow the conventions of [17] for 11–dimensional Supergravity and of [18, 9, 19, 31] for its Kaluza Klein compactifications. In $D = 11$ the flat metric is mostly minus

$$\eta_{AB} = (+, \underbrace{-, \ldots, -}_{10 \text{ times}})$$

so that the flat metric for the internal 7–dimensional geometry is purely negative $\eta_{\alpha\beta} = -\delta_{\alpha\beta}$. Since we use differential form language the only indices we introduce are the flat ones $(A, B, C, \ldots, = 0, 8, 9, 10, 1, 2, \ldots, 7)$. We also split them into:

$$A = \underbrace{a}_{0,8,9,10}, \underbrace{\alpha}_{1,\ldots,7}.$$

With the negative metric the gamma matrices in 7–dimensions are purely real and antisymmetric and are named τ_α:

$$\{\tau_\alpha\,,\,\tau_\beta\} = -\delta_{\alpha\beta}$$

B Explicit form of the harmonics

In this Appendix, we give some discussions and expressions regarding the harmonics of the "super-Higgs" shadow gravitino multiplet which were not included in section 3.2. We refer to table 4 for notations.

· *The harmonic of the massive π scalars with $E = 5$* When we discussed the mechanism of shadowing we announced that it originates from two general features:

1. The fact that the same harmonic is associated with two different Kaluza Klein fields
2. The fact that, via Killing spinor multiplication, each Bose/Fermi harmonic generates a new Fermi/Bose harmonic with predetermined eigenvalue.

We have already taken advantage of the second type of *shadowing* when we have constructed the harmonic (3.43) of the Z–vectors. Let us use it once again to show that the shadow of the massless gravitinos contains also the pseudoscalars π with scale dimension $E = 5$. It suffices to use eq.s (4.47b),(4.49) and (4.50) of [9] that instruct us how to construct an eigenstate of the operator (3.11) with eigenvalue:

$$M_{(1)^3} = -\tfrac{1}{4}\left(M_{(\frac{1}{2})^3} + 8\right) \tag{B.1}$$

from each eigenstate of the internal Dirac operator (3.13) of eigenvalue $M_{(\frac{1}{2})^3}$. From the massless gravitino harmonics ($M_{(\frac{1}{2})^3} = -16$), via such a relation, we obtain the harmonics (3.46), namely:

$$\Upsilon^{(AB)}_{\alpha\beta\gamma} = \overline{\eta}^{A}\,\tau_{\alpha\beta\gamma}\,\eta^{B} \tag{B.2}$$

which are bilinear in Killing spinors and belong to the eigenvalue $M_{(1)^3} = -2$ of (3.13). These are the harmonics of the pseudoscalar particles of energy $E = 5$ and isospin $J = 2 \oplus J = 0$ displayed in table (3). Indeed the three index τ–matrix is symmetric so that we have symmetry in the SO(3) R–symmetry indices A, B. Decomposing into the traceless and trace parts we obtain the $\mathbf{J} = 2$ and $\mathbf{J} = 0$ states.

· *The harmonic of the massive $E = 9/2$ spinors* We can now return to the $E = \frac{9}{2}$ spinors we have left aside. Their harmonic is of the transverse type Ξ_α and must be an eigenvalue of the operator (3.14) with eigenvalue $M_{\frac{3}{2}(\frac{1}{2})^2} = 4$. Recalling eq.s (5.7-5.12) of [9] we see that, with suitable coefficients a, b, c the fermionic combination:

$$\Omega_\alpha = a\,\tau_{\alpha\mu\nu\rho}\eta\,Y_{\mu\nu\rho} + b\,\tau_{\mu\nu}\eta\,Y_{\alpha\mu\nu} + c\,\tau_{\mu\nu\rho}\eta\,D_\alpha Y_{\mu\nu\rho} \tag{B.3}$$

has the desired eigenvalue

$$4 \,=\, M_{\frac{3}{2}(\frac{1}{2})^2} = -4\left(M_{(1)^3} + 4\right) \tag{B.4}$$

if $Y_{\alpha\beta\gamma}$ is identified with the bosonic harmonic (3.46) of eigenvalue $M_{(1)^3} = -2$. Inserting the values of a, b, c given in eq.s (5.9a-5.9b) of [9] we consider therefore the following $\mathrm{SO}(3)_R$ 3-index tensor which is also an SO(7) τ-traceless spinor-vector

$$\begin{aligned}\Omega^{A(BC)}_\alpha &= 14\,\tau_{\alpha\mu\nu\rho}\,\eta^A\,\overline{\eta}^B\,\tau^{\mu\nu\rho}\,\eta^C \;+\; 48\,\tau^{\mu\nu}\,\eta^A\,\overline{\eta}^B\,\tau_{\alpha\mu\nu}\eta^C \\ & \quad 2\,\tau^{\mu\nu\rho}\,\eta^A\,\overline{\eta}^B\,\tau_{\alpha\mu\nu\rho}\,\eta^C\end{aligned} \tag{B.5}$$

A priori we are multiplying an isospin $J = 1$ with $J = 2$ or $J = 0$, so that we can obtain $J = 3$, $J = 2$, $J = 1$ and $J = 1$ a second time, according to the numerology and $7 + 5 + 3 + 3 = 18 = 3 \times (5 + 1)$. However Fierz identities on the η–spinors should imply that we get only an isospin J=2 state and an isospin J=1 state since this is what is required by the structure of the AdS_4 spin 3/2 massive multiplet displayed in table 4 where no $J = 3$ does appear. This is completely verified by explicit calculations that were numerically performed introducing an explicit basis for τ matrices and choosing Killing spinors η^A in three arbitrary directions. The projection on the $J = 3$ state corresponds to full symmetrization in the indices (ABC) and then removal of the trace: the result of this projection is identically zero. Hence it turns out that, in the $\mathrm{SO}(3)_R$ indices the tensor $\Omega^{A(BC)}$ has automatically the following symmetry:

$$\begin{array}{|c|c|}\hline B & C \\ \hline A & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{B.6}$$

We obtain the isospin $J = 2$ projection by defining

$$\widehat{\Omega}^{(XY)}_\alpha = \tfrac{1}{2}\,\left(\epsilon^{XTZ}\,\Omega^{T(ZY)}_\alpha \;+\; \epsilon^{YTZ}\,\Omega^{T(ZX)}_\alpha\right) \tag{B.7}$$

while we obtain the isospin $J = 1$ projection by defining

$$\widehat{\Omega}^{(A)}_\alpha = \sum_{B=1}^{3} \Omega^{A(BB)}_\alpha \tag{B.8}$$

Indeed, because of the symmetry (B.6) the symmetric tensor $\widehat{\Omega}^{(XY)}_\alpha$ is automatically traceless in the indices XY and hence a pure $J = 2$ representation. This can also be numerically verified.

· *The harmonic of the massive $E = 7/2$ spinors* Also these states have a harmonic of the transverse type Ξ_α. The eigenvalue, this time is $M_{\frac{3}{2}(\frac{1}{2})^2} = -16$. Looking at fig.1, we see that the transverse spinor harmonics communicate either with the 1–forms Y_α or with the 2–forms $Y_{\alpha\beta}$ or with the 3–forms $Y_{\alpha\beta\gamma}$ or, finally with the symmetric tensors $Y_{(\alpha\beta)}$. We are descending the multiplet from higher to lower spins since we want to express everything in terms of Killing spinors using the harmonics we have so far already constructed. Hence the mass relation which is relevant to us at this point is that given in eq. (4.70) of [9], namely:

$$M_{(1)^2 0} = (M_{\frac{3}{2}(\frac{1}{2})^2} + 8)(M_{\frac{3}{2}(\frac{1}{2})^2} + 4) \tag{B.9}$$

Indeed it is satisfied by $M_{(1)^2 0} = 96$ and $M_{\frac{3}{2}(\frac{1}{2})^2} = -16$, which shows that in terms of a harmonic Θ^A_α which is eigenstate of the operator (3.14) with eigenvalue $M_{\frac{3}{2}(\frac{1}{2})^2} = -16$ we could construct the harmonic (3.43) of the $E = 4, J = 1$ vector fields of type Z. Such a relation was given in eq.s (4.67b-4.69) of [9]. We are interested in the inverse relation which expresses Θ^A_α in terms of the two form (3.43) and of Killing spinors. Such an inverse relation was not given in [9] but it can be easily derived. In our explicit case we find eq.(3.47), namely:

$$\Theta^A_\alpha \equiv \epsilon^{ABC} \left(\frac{3}{16} \tau_{\alpha\mu\nu\rho} \, \eta^B \, D_\mu \, T^C_{\nu\rho} + \frac{9}{2} \tau_\mu \, \eta^B \, T^C_{\alpha\mu} \right) \tag{B.10}$$

So also this harmonic is universal and depends only on Killing spinors

C The $\mathrm{SU}(3,3)/\mathrm{SU}(3) \times \mathrm{SU}(3) \times U(1)$ solvable Lie algebra

For the reader's convenience we report in this appendix the structure of the solvable Lie algebra associated with the manifold $\mathrm{SU}(3,3)/\mathrm{SU}(3)\times\mathrm{SU}(3)\times U(1)$ that was already presented in [55]. Since the coset manifold $\mathrm{SU}(3,3)/\mathrm{SU}(3)\times U(3)$ is a *Special Kähler* manifold the elements of *Solv* can be described in the Alekseevski's formalism for Kählerian algebras [61] whose general structure is given by the following general algebraic relations:

$$\begin{aligned}
\mathrm{Solv} &= F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \\
F_i &= \{\mathrm{h}_i \, , \, \mathrm{g}_i\} \quad i = 1,2,3 \\
\mathbf{X} &= \mathbf{X}^+ \oplus \mathbf{X}^- \\
\mathbf{Y} &= \mathbf{Y}^+ \oplus \mathbf{Y}^- \\
\mathbf{Z} &= \mathbf{Z}^+ \oplus \mathbf{Z}^- \\
[\mathrm{h}_i \, , \, \mathrm{g}_i] &= 2\mathrm{g}_i \quad i = 1,2,3 \\
[F_i \, , \, F_j] &= 0 \quad i \neq j \\
\left[\mathrm{h}_3 \, , \, \mathbf{Y}^\pm\right] &= \pm \mathbf{Y}^\pm \\
\left[\mathrm{h}_3 \, , \, \mathbf{X}^\pm\right] &= \pm \mathbf{X}^\pm \\
\left[\mathrm{h}_2 \, , \, \mathbf{Z}^\pm\right] &= \pm \mathbf{Z}^\pm \\
\left[\mathrm{h}_1 \, , \, \mathbf{Z}^\pm\right] &= \mathbf{Z}^\pm \\
\left[\mathrm{h}_1 \, , \, \mathbf{Y}^\pm\right] &= \mathbf{Y}^\pm \\
[\mathrm{g}_1 \, , \, \mathbf{Y}] &= [\mathrm{g}_1 \, , \, \mathbf{Z}] = 0 \\
\left[\mathrm{g}_3 \, , \, \mathbf{Y}^+\right] &= \left[\mathrm{g}_2 \, , \, \mathbf{Z}^+\right] = \left[\mathrm{g}_3 \, , \, \mathbf{X}^+\right] = 0 \\
\left[\mathrm{g}_3 \, , \, \mathbf{Y}^-\right] &= \mathbf{Y}^+ ; \; \left[\mathrm{g}_2 \, , \, \mathbf{Z}^-\right] = \mathbf{Z}^+ ; \; \left[\mathrm{g}_3 \, , \, \mathbf{X}^-\right] = \mathbf{X}^+ \\
[F_1 \, , \, \mathbf{X}] &= [F_2 \, , \, \mathbf{Y}] = [F_3 \, , \, \mathbf{Z}] = 0 \\
\left[\mathbf{X}^- \, , \, \mathbf{Z}^-\right] &= \mathbf{Y}^-
\end{aligned} \tag{C.1}$$

Explicitly the corresponding SU$(3,3)$ matrices are listed below [17] :

$$
\mathrm{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad
\mathrm{g}_1 = \begin{pmatrix} \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathrm{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad
\mathrm{g}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathrm{h}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad
\mathrm{g}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \end{pmatrix}
$$

$$
\mathbf{X}_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \end{pmatrix} \quad
\mathbf{X}_2^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}
$$

$$
\mathbf{X}_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & \frac{-i}{2} & 0 & 0 & \frac{i}{2} & 0 \end{pmatrix} \quad
\mathbf{X}_2^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}
$$

$$
\mathbf{Y}_1^+ = \begin{pmatrix} 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \end{pmatrix} \quad
\mathbf{Y}_2^+ = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}
$$

[17] We want to express our gratitude to Mario Trigiante for providing us with this explicit representation of the generators and for his invaluable advice on this algebraic point

$$
\begin{aligned}
\mathbf{Y}_1^- &= \begin{pmatrix} 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{2} & 0 & 0 & \frac{i}{2} & 0 & 0 \end{pmatrix} &
\mathbf{Y}_2^- &= \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \\
\mathbf{Z}_1^+ &= \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} &
\mathbf{Z}_2^+ &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{Z}_1^- &= \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & \frac{i}{2} & 0 \\ \frac{-i}{2} & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} &
\mathbf{Z}_2^- &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{C.2}
$$

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